Conditional regularity for an elastic shell interacting with the Navier-Stokes equations

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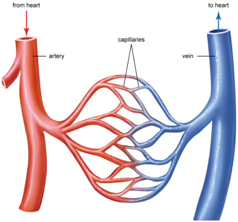
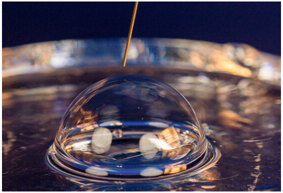
1. Motivation

Examples for fluid-elastic structure interaction (FSI):





- Modeling of fluid and structure:

Fluid: incomp./comp.; inviscid/viscous; homoge./inhomoge.;
Structure: elastic; plate/shell; with/without dissipation;

- Boundary conditions: periodic/clamped boundary;
- Fluid-structure coupling:

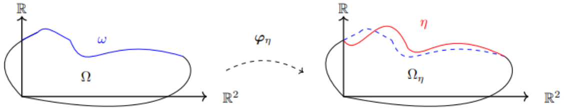
action-reaction principle; balance of forces on the interface.

- Long-term running: small deformation; no intersection (contact);

2. Problem setting

- Assumptions:

The beam located on the top of the 2D/3D canister is elastic;
The fluid is viscous incompressible;



- The beam deformation $\eta : (t, \mathbf{y}) \in I \times \omega \mapsto \eta(t, \mathbf{y}) \in \mathbb{R}$;
- The reference domain Ω and the deformed domain Ω_η.
- $\omega \subset \mathbb{R}^2$: The flexible part of the boundary;
- $\varphi_\eta : \omega \to \partial\Omega_\eta$ that parametrizes the boundary of the reference domain Ω.
- The velocity field $\mathbf{v} : (t, \mathbf{x}) \in I \times \Omega_\eta \mapsto \mathbf{v}(t, \mathbf{x}) \in \mathbb{R}^3$;

The pressure function $\pi : (t, \mathbf{x}) \in I \times \Omega_\eta \mapsto \pi(t, \mathbf{x}) \in \mathbb{R}$.

2.1 Governing equations

- Navier-Stokes equations for the fluid:

$$\begin{cases} \varrho_f\big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\big) = \mu\Delta\mathbf{v} - \nabla\pi & (t,\mathbf{x}) \in I\times\Omega_\eta, \\ \operatorname{div}\mathbf{v} = 0 & (t,\mathbf{x}) \in I\times\Omega_\eta, \\ \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \mathbf{x}\in\Omega_{\eta_0}, \end{cases} \tag{1}$$

- linearly elastic beam equation, $(t,\mathbf{y}) \in I\times\omega$,

$$\begin{cases} \varrho_s\partial_t^2\eta - \gamma\partial_t\Delta_{\mathbf{y}}\eta + \alpha\Delta_{\mathbf{y}}^2\eta = \phi(\mathbf{v},\pi,\eta)(t,\mathbf{y}), \\ \eta(0,\mathbf{y}) = \eta_0(\mathbf{y}), \quad (\partial_t\eta)(0,\mathbf{y}) = \eta_*(\mathbf{y}). \end{cases} \tag{2}$$

- Interface coupling (kinematic and dynamic conditions):

$$\mathbf{v}\circ\boldsymbol{\varphi}_\eta = \partial_t\eta\mathbf{n} \quad \text{on } I\times\omega, \tag{3}$$

$$\phi(\mathbf{v},\pi,\eta)(t,\mathbf{y}) = -\mathbf{n}^\intercal\tau\circ\boldsymbol{\varphi}_\eta\mathbf{n}_\eta|\det(\nabla_{\mathbf{y}}\boldsymbol{\varphi}_\eta)| \quad \text{on } I\times\omega. \tag{4}$$

- No-slip at the fixed part and clamped at $\partial\omega$:

$$\mathbf{v} = 0, \quad \text{on } I\times(\partial\Omega\setminus\omega);$$

$$\eta = 0, \quad \nabla_{\mathbf{y}}\eta = 0 \quad \text{on } \partial\omega.$$

2.2 Deformation principle

▷ For shell: The shell deforms normally

Parameterization $\boldsymbol{\varphi} : \omega \to \partial\Omega$; $\quad \boldsymbol{\varphi}_\eta : \omega \to \partial\Omega_\eta$;

$$\boldsymbol{\varphi}_\eta(t, \mathbf{y}) = \boldsymbol{\varphi}(\mathbf{y}) + \eta(t, \mathbf{y})\mathbf{n}(\mathbf{y}), \quad \mathbf{y} \in \omega,\, t \in I.$$

Hanzawa transform $\boldsymbol{\Psi}_\eta : \Omega \to \Omega_\eta$ defined by

$$\boldsymbol{\Psi}_\eta(\mathbf{x}) = \begin{cases} \mathbf{p}(\mathbf{x}) + \big(s(\mathbf{x}) + \eta(\mathbf{y}(\mathbf{x}))\phi(s(\mathbf{x}))\big)\mathbf{n}(\mathbf{y}(\mathbf{x})) & \text{dist}(\mathbf{x}, \partial\Omega) < L, \\ \mathbf{x} & \text{elsewhere.} \end{cases}$$

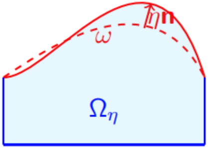
$$s(\mathbf{x}) \in S_L := \{\mathbf{x} \in \mathbb{R}^n : \text{dist}(\mathbf{x}, \partial\Omega) < L\}.$$

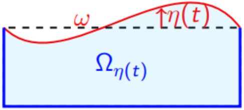
For $\|\eta\|_{L^\infty} < \alpha < L$, cutoff $\phi \in C^\infty(\mathbb{R})$, $|\phi'| < L/\alpha$, $\boldsymbol{\Psi}_\eta^{-1}$ exists.

▷ For plate: The plate deforms vertically

Explicit transform:

$$\chi : \Omega \to \Omega_\eta$$
$$(x, z) \mapsto (x, (1+\eta)z).$$





2.3 Related work (incompressible)

▷ For plate:

- Grandmont '08: weak solution 3D for elastic plate;
- Lequeurre '11: local strong 2D for damped plate;
- Grandmont, Hillairet '16: global strong 2D for damped plate;
- Grandmont, Hillairet, Lequeurre '19:
  local strong 2D for damped plate and wave case;
- Badra, Takahashi: '19,'21: elastic case and wave with small data;
  '22: elastic with regular data;
- Schwarzacher, Sroczinski '22: Serrin condition for plate;
- Schwarzacher, Su '23: 2D strong for elastic plate with large, less regular data.

▷ For shell:

- Muha, Canic '13: Weak solution for 2D-1D coupling;
- Lengeler, Růžička '14: weak solution 3D;
- Muha, Schwarzacher '22: additional regularity 3D;
- Breit, Mensah, Schwarzacher, Su '23: Serrin condition 3D;

- Transform to the reference domain: $\overline{\pi} = \pi \circ \mathbf{\Psi}_\eta, \quad \overline{\mathbf{v}} = \mathbf{v} \circ \mathbf{\Psi}_\eta$

$$\begin{aligned}
&\mathbf{B}_{\eta 0} : \nabla \overline{\mathbf{v}} = h_\eta(\overline{\mathbf{v}}), \\
&\partial_t^2 \eta - \partial_t \Delta_{\mathbf{y}} \eta + \Delta_{\mathbf{y}}^2 \eta = \mathbf{n}^\intercal \left[\mathbf{H}_\eta(\overline{\mathbf{v}}, \overline{\pi}) - \mathbf{A}_{\eta 0} \nabla \overline{\mathbf{v}} + \mathbf{B}_{\eta 0} \overline{\pi} \right] \circ \boldsymbol{\varphi} \mathbf{n}, \\
&J_{\eta 0} \partial_t \overline{\mathbf{v}} - \mathrm{div}(\mathbf{A}_{\eta 0} \nabla \overline{\mathbf{v}}) + \mathrm{div}(\mathbf{B}_{\eta 0} \overline{\pi}) = \mathbf{h}_\eta(\overline{\mathbf{v}}) - \mathrm{div} \mathbf{H}_\eta(\overline{\mathbf{v}}, \overline{\pi})
\end{aligned}$$

- Compatibility condition

Incompressibility, kinematic, no-slip, clamped conditions

$$\Longrightarrow \int_\omega \partial_t \eta \mathrm{d}\mathbf{x} = 0 \quad \overset{\text{beam equation}}{\Longrightarrow} \quad \int_\omega \phi(\mathbf{v}, \pi, \eta) \mathrm{d}\mathbf{x} = 0.$$

Initial data:

$$\begin{aligned}
&\mathrm{div}\, \mathbf{v}_0 = 0, \quad \mathbf{v}_0 = 0 \quad \text{on } \partial\Omega \setminus \omega, \\
&\mathbf{v}_0 \circ \boldsymbol{\varphi}_{\eta 0} = \eta_*(\mathbf{y}) \mathbf{n}, \\
&\int_\omega \eta_*(\mathbf{y}) \mathrm{d}\mathbf{y} = 0, \quad \text{no intersection at } t = 0.
\end{aligned} \tag{5}$$

For the strong solution we assume that

$$\eta_0 \in W^{3,2}(\omega), \quad \eta_* \in W^{1,2}(\omega), \quad \mathbf{v}_0 \in W^{1,2}_{\mathrm{div}}(\Omega_{\eta 0}), \tag{6}$$

3. Local strong solution

- Linearized system

For given $(h, \mathbf{h}, \mathbf{H})$ such that

$$h \in L^2\big(I; W^{1,2}(\Omega)\big) \cap W^{1,2}\big(I; W^{-1,2}(\Omega)\big) \cap \{h(0,\mathbf{x}) = 0\},$$
$$\mathbf{h} \in L^2(I \times \Omega), \quad \mathbf{H} \in L^2(I; W^{1,2}(\Omega)),$$

Proposition 1. Under the assumption for $(\eta_0, \eta_*, \overline{\mathbf{v}}_0, h, \mathbf{h}, \mathbf{H})$, there exists a strong solution for linearized FSI system.

- 4.3 Fixed-point argument

$$X_{I_*} := (\zeta, \overline{\mathbf{w}}, \overline{q}) \in$$
$$\big(W^{1,\infty}\big(I_*; W^{1,2}(\omega)\big) \cap L^{\infty}\big(I_*; W^{3,2}(\omega)\big) \cap W^{1,2}\big(I_*; W^{2,2}(\omega)\big) \cap W^{2,2}\big(I_*; L^2(\omega)\big)\big)$$
$$\times \big(L^{\infty}\big(I_*; W^{1,2}(\Omega)\big) \cap W^{1,2}\big(I_*; L^2(\Omega)\big) \cap L^2\big(I_*; W^{2,2}(\Omega)\big)\big) \times L^2\big(I_*; W^{1,2}(\Omega)\big),$$

Theorem 2. There exists a time $T > 0$ and $R > 0$, s.t. the map $\mathcal{T}$

$$\mathcal{T} : B_R^{X_{I_*}} \to B_R^{X_{I_*}}$$
$$(\zeta, \overline{\mathbf{w}}, \overline{q}) \mapsto (\eta, \overline{\mathbf{v}}, \overline{\pi}),$$

is a contraction map, which thereby possesses a fixed point in X_{I_*}

- Energy estimate

$$\sup_{I^*} \|\mathbf{v}\|_{L^2_{\mathbf{x}}}^2 + \int_{I^*} \|\nabla \mathbf{v}\|_{L^2_{\mathbf{x}}}^2 \,\mathrm{d}t \lesssim C_0,$$

$$\sup_{I^*} \|\partial_t \eta\|_{L^2_{\mathbf{y}}}^2 + \sup_{I^*} \|\Delta_{\mathbf{y}} \eta\|_{L^2_{\mathbf{y}}}^2 + \int_{I^*} \|\partial_t \nabla_{\mathbf{y}} \eta\|_{L^2_{\mathbf{y}}}^2 \,\mathrm{d}t \lesssim C_0$$

Proposition 3. Let $(\eta_0, \eta_*, \mathbf{v}_0)$ satisfy (5) and (6). Suppose that $(\eta, \mathbf{v})$ is a strong solution to (1)–(4). For $r \in [2, \infty)$, $s \in (3, \infty]$

$$\mathbf{v} \in L^r(I; L^s(\Omega_\eta)), \quad \frac{2}{r} + \frac{3}{s} \leqslant 1, \quad \eta \in L^\infty(I; C^{0,1}(\omega)).$$

Then

$$\sup_{I_*} \int_\omega \left(|\partial_t \nabla_{\mathbf{y}} \eta|^2 + |\nabla_{\mathbf{y}} \Delta_{\mathbf{y}} \eta|^2\right) \mathrm{d}\mathbf{y} + \sup_{I_*} \int_{\Omega_\eta} |\nabla \mathbf{v}|^2 \,\mathrm{d}\mathbf{x}$$
$$+ \int_{I_*} \int_\omega \left(|\partial_t \Delta_{\mathbf{y}} \eta|^2 + |\partial_t^2 \eta|^2\right) \mathrm{d}\mathbf{y}\,\mathrm{d}t + \int_{I_*} \int_{\Omega_\eta} \left(|\nabla^2 \mathbf{v}|^2 + |\partial_t \mathbf{v}|^2 + |\nabla \pi|^2\right) \mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$\lesssim \int_\omega \left(|\nabla_{\mathbf{y}} \eta_*|^2 + |\nabla_{\mathbf{y}} \Delta_{\mathbf{y}} \eta_0|^2\right) \mathrm{d}\mathbf{y} + \int_{\Omega_{\eta_0}} |\nabla \mathbf{v}_0|^2 \,\mathrm{d}\mathbf{x},$$

5. Weak-strong uniqueness

$(\mathbf{v}_1, \eta_1)$: a weak solution and $(\mathbf{v}_2, \eta_2)$: a strong solution.
Transform the strong solution in the weaker domain:

$$\mathbf{\Psi}_{\eta_2 - \eta_1} : \Omega_{\eta_1} \to \Omega_{\eta_2}$$

$$\underline{\mathbf{v}}_2 := \mathbf{v}_2 \circ \mathbf{\Psi}_{\eta_2-\eta_1}, \quad \underline{\pi}_2 := \pi_2 \circ \mathbf{\Psi}_{\eta_2-\eta_1}, \quad \underline{\mathbf{f}}_2 := \mathbf{f}_2 \circ \mathbf{\Psi}_{\eta_2-\eta_1},$$

Theorem 4. Suppose further that

$$\eta_1 \in L^\infty(I; C^{0,1}(\omega)).$$

Then we have

$$\begin{aligned}
&\sup_{t\in I} \int_{\Omega_{\eta_1(t)}} |\mathbf{v}_1(t) - \underline{\mathbf{v}}_2(t)|^2 \,\mathrm{d}\mathbf{x} + \sup_{t\in I} \int_\omega \left(|\partial_t(\eta_1 - \eta_2)(t)|^2 + |\Delta_{\mathbf{y}}(\eta_1 - \eta_2)(t)|^2 \right) \mathrm{d}\mathbf{y} \\
&+ \int_I \int_{\Omega_{\eta_1(\sigma)}} |\nabla(\mathbf{v}_1 - \underline{\mathbf{v}}_2)|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_I \int_\omega |\partial_t \nabla_{\mathbf{y}}(\eta_1 - \eta_2)|^2 \,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\lesssim \int_{\Omega_{\eta_{0,1}}} |\mathbf{v}_1(0) - \underline{\mathbf{v}}_2(0)|^2 \,\mathrm{d}\mathbf{x} + \int_\omega |\partial_t(\eta_1 - \eta_2)(0)|^2 \,\mathrm{d}\mathbf{y} + \int_\omega |\Delta_{\mathbf{y}}(\eta_1 - \eta_2)(0)|^2 \,\mathrm{d}\mathbf{y}
\end{aligned}$$

5. Weak-strong uniqueness

Theorem 5. Let $T > 0$ be given. Let $(\mathbf{v}, \eta)$ be a weak solution of (1)–(4). Suppose that we have

$$\mathbf{v} \in L^r(I; L^s(\Omega_\eta)), \quad \frac{2}{r} + \frac{3}{s} \leqslant 1,$$
$$\eta \in L^\infty(I; C^{0,1}(\omega)).$$

Then $(\mathbf{v}, \eta)$ is a strong solution on $I = (0, t)$, where $t < T$ only in case $\Omega_{\eta(s)}$ approaches a self-intersection when $s \to t$. Moreover, $(\mathbf{v}, \eta)$ is unique in the class of weak solutions with deformation in $L^\infty(I, C^{0,1}(\omega))$.

- Local strong on $(0, T^*)$;
- With Lipschitz condition, $(\mathbf{v}, \eta)$ is a strong solution;
- With Serrin condition together, apply acceleration estimate;
- Go back to the local existence to obtain the solution on $(T^*, 2T^*)$,...

[1] D. Breit, P. Mensah, S. Schwarzacher and P. Su,
Ladyzhenskaya-Prodi-Serrin condition for fluid-structure interaction systems,
arXiv: 2307.12273, 2023.

6. Discussion and remark

1. Does the dissipation of the shell make difference?
   - Acceleration estimate;
   - Global existence ?

3. Lipschitz condition for η:
   - Stokes estimate; Bogovskij operator;
   - Equivalence through $\mathbf{\Psi}_\eta$ ($W^{2,2} \hookrightarrow C^{0,1}$ fails).

2. What is the difference between shell and plate?
   - Explicit expression for plate;
   - The regularity issue in the compressible case;

3. The borderline case $s = 3$ and $r = \infty$, i.e. $\mathbf{v} \in L^\infty(I; L^3(\Omega_\eta))$?

   - For Navier-Stokes: (Masuda '84, Escauriaza et al. '03);
   - For fluid-rigid body system by Maity and Takahashi '23;

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Thanks for your attention !

https://sites.google.com/view/peisu

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