Cahn-Hillard and Keller-Segel systems as high-friction limits of gas dynamics

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High-friction limit

- The high-friction limit (relaxation limit) is a part of a long research program of establishing a connection between nonlinear hyperbolic systems and degenerate diffusion equations.
- One of the first results in this direction (Marcati Milani 1990) states that the solutions to the compressible Euler equations in $1d$

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho\, u) = 0, \\ &\varepsilon^2 \partial_t(\rho\, u) + \partial_x(\varepsilon^2\, \rho\, u^2 + p(\rho)) = -u \end{aligned}$$

converge, as $\varepsilon \to 0$, to the porous media equation

$$\partial_t \rho = \partial_x \left(\rho\, \partial_x p(\rho)\right)$$

where $p(\rho)$ is the pressure function of the form $p(\rho) = \rho^\gamma$.

High friction limit

- Let us look at the rescaled system

$$
\begin{aligned}
\partial_t \rho + \frac{1}{\varepsilon} \partial_x (\rho\, u) &= 0, \\
\partial_t (\rho\, u) + \frac{1}{\varepsilon}\, \partial_x (\, \rho\, u^2 + p(\rho)) &= -\frac{u}{\varepsilon^2}.
\end{aligned}
$$

- Intuitively, it is easy to understand that the flow of the fluid with big damping or friction (caused by the term $-\frac{u}{\varepsilon^2}$) and very small kinetic energy (caused by the initial condition) resembles a flow through a porous media.
- The revival of interest in this type of problem appeared recently with an observation that one can study these problems by the relative entropy method - often used in the context of weak-strong uniqueness.

Relative entropy method - some remarks

- Important assumption: initial datum is well-prepared: initial velocity $\mathbf{u}_\varepsilon^0$ vanishes as the parameter $\varepsilon \to 0$ so that the initial kinetic energy is very small.
- Then the relative entropy at time $t = 0$ converges to 0 as $\varepsilon \to 0$.
- Solutions to target system need to be regular.
- One can study similar problems via compactness methods and this approach is also effective for ill-prepared initial data. Nevertheless, its applicability is restricted to some special cases like $1d$ or the presence of viscosity terms yielding compactness.

Diffusive relaxation limits

Euler, Euler-Poisson, Euler-Korteweg systems with friction

$$
\begin{aligned}
\partial_t \rho + \operatorname{div}(\rho u) &= 0, \\
\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + k\rho u &= -a\nabla\rho^\gamma + b\rho\nabla\Phi_\rho + c\rho\nabla(\Delta\rho), \\
-\Delta\Phi_\rho &= \rho - M_\rho
\end{aligned}
$$

for $a > 0$, $c \geq 0$, $b \in \mathbb{R}$

High friction limit $k \to \infty$

For large frictions $k = \frac{1}{\varepsilon}$, and a proper scaling of time $\partial_t \mapsto \varepsilon \partial_t$

$$\partial_t \rho + \frac{1}{\varepsilon} \operatorname{div}(\rho u) = 0,$$
$$\partial_t(\rho u) + \frac{1}{\varepsilon} \operatorname{div}(\rho u \otimes u) + \frac{1}{\varepsilon} \nabla \rho^\gamma = -\frac{1}{\varepsilon^2} \rho u,$$

(compressible Euler)

As ε goes to zero:

$$\partial_t \rho + \Delta(\rho)^\gamma = 0,$$

(porous media equation)

Other contributions: F. Huang, P. Marcati, R. Pan, A. J. Milani, R. Natalini

C. Lattanzio and A. E. Tzavaras. From gas dynamics with large friction to gradient flows describing diffusion theories. Comm. PDE, 2017.

$$\partial_t \rho + \frac{1}{\varepsilon}\operatorname{div}(\rho u) = 0,$$
$$\partial_t(\rho u) + \frac{1}{\varepsilon}\operatorname{div}(\rho u \otimes u) + \frac{1}{\varepsilon^2}\rho u = -\frac{a}{\varepsilon}\nabla \rho^\gamma + \frac{b}{\varepsilon}\rho \nabla \Phi,$$
$$-\Delta \Phi = \rho - M_\rho$$

(Euler-Poisson)

As ε goes to zero:

$$\partial_t \rho + \operatorname{div}(-a\nabla \rho^\gamma + b\rho\nabla\Phi) = 0,$$
$$-\Delta \Phi = \rho - M_\rho.$$

(parabolic–elliptic Keller–Segel)

where M_ρ denotes the spatial average over the density.

C. Lattanzio and A. E. Tzavaras. From gas dynamics with large friction to gradient flows describing diffusion theories. Comm. PDE, 2017.

$$\partial_t \rho + \frac{1}{\varepsilon}\operatorname{div}(\rho u) = 0,$$

$$\partial_t(\rho u) + \frac{1}{\varepsilon}\operatorname{div}(\rho u \otimes u) + \frac{1}{\varepsilon^2}\rho u = -\frac{a}{\varepsilon}\nabla \rho^\gamma + \frac{1}{\varepsilon}\rho \nabla \Delta \rho,$$

(Euler–Korteweg)

As ε goes to zero:

$$\partial_t \rho + \operatorname{div}(-a\nabla \rho^\gamma + \rho \nabla \Delta \rho) = 0,$$

(singular Cahn–Hillard)

Common formulation

We combine them into one system, which we call Euler–Korteweg–Poisson system (EKP)

$$
\begin{aligned}
\partial_t \rho + \tfrac{1}{\varepsilon}\operatorname{div}(\rho u) &= 0, \\
\partial_t(\rho u) + \tfrac{1}{\varepsilon}\operatorname{div}(\rho u \otimes u) + \tfrac{1}{\varepsilon^2}\rho u &= -\tfrac{a}{\varepsilon}\nabla \rho^\gamma + \tfrac{b}{\varepsilon}\rho \nabla \Phi_\rho + \tfrac{c}{\varepsilon}\rho \nabla(\Delta \rho), \\
-\Delta \Phi_\rho &= \rho - M_\rho
\end{aligned}
$$

for $a, \varepsilon > 0$, $c \geq 0$, $b \in \mathbb{R}$

- To prove the convergence, we use the relative entropy method.
- The method is based on introducing a functional called relative entropy, which measures the dissipation between two solutions of the system.
- The studies of Lattanzio and Tzavaras rely on direct calculation of the relative entropy between a weak entropy dissipative solution and a smooth, entropy conservative solution for the thermomechanical process.

Formal/rigorous result – Euler-Poisson

- **Weak solutions** to Euler-Poisson with attractive potential ($b < 0$) converge to **strong solutions** of Keller-Segel equation,
- Existence of weak solutions to Euler-Poisson?
- Are any weak solutions good?
  - D. Donatelli, E. Feireisl, P. Marcati, Well/ill posedness for the Euler-Korteweg-Poisson system and related problems. *Commun. PDE*, 2015
    - these solutions will have a jump in the energy at the initial time so they will not be dissipative (admissible).

Weak solutions

Starting point: existence of admissible (satisfying energy inequality) weak solutions of the first system.

Problem: weak solutions existing on arbitrary intervals of time are not known to exist for most models in fluids dynamics.

Approach: the concept of dissipative measure-valued solutions. While measure-valued solutions are weaker than the usual weak solutions, they are dissipative and they are known to exist.

In the direction of rigorous asymptotics

- Contrary to above mentioned papers we want to consider such limits on the level of measure-valued solutions, since it is not clear that weak (global in time) solutions exist.

The existence of dissipative measure-valued solutions to all the cases can be shown, but the question of existence of strong solutions is a more complex matter. One can show:

- global existence of strong solutions to chemo-repulsive Keller–Segel and porous media equations,
- local existence of strong solutions to chemo-attractive Keller–Segel equations,
- global existence of strong solutions for small initial data to Cahn–Hillard equations.

Strong solution to target system

Definition

We will say that (r, Φ_r) is a strong solution to the system

$$\partial_t \rho - \mathrm{div}(a\nabla_x(\rho^\gamma) + b\rho\nabla\Phi_\rho - c\rho\nabla(\Delta\rho)) = 0,$$
$$-\Delta\Phi_\rho = \rho - M_\rho.$$

if

- $(r, \Phi_r) \in C^{5,1}(\mathbb{T}^d \times [0, T])$, whenever $c > 0$,
- $(r, \Phi_r) \in C^{2,1}(\mathbb{T}^d \times [0, T])$, whenever $c = 0$.

Relative entropy method for hydrodynamics

Measure-valued strong uniqueness to compressible Euler and Navier-Stokes like systems

- P. Gwiazda, A. ŚG, E. Wiedemann, *Weak-Strong Uniqueness for Measure-Valued Solutions of Some Compressible Fluid Models*, Nonlinearity, 2015
- E. Feireisl, P. Gwiazda, A. ŚG, E. Wiedemann, *Dissipative measure-valued solutions to the compressible Navier-Stokes system*, Calc. Var. and PDE, 2016
- J. A. Carrillo, T. Dębiec, P. Gwiazda, A. ŚG, *Dissipative measure-valued solutions to the Euler-Poisson equation*, SIAM J. Math. Anal., 2024

Measure-valued solutions

- Often times one cannot guarantee the L^p bound of the sequence with $p > 1$ and the concentrations might appear.
- Suppose that f is a continuous function and $\{f(z_j)\}_{j\in\mathbb{N}}$ is bounded in $L^\infty((0,T);L^1(\Omega))$, then there exists $m^f \in L^\infty((0,T);\mathcal{M}(\Omega))$, such that

$$f(z_j) - \langle \nu_{t,x}, f\rangle \overset{*}{\rightharpoonup} m^f \qquad \text{weakly-}* \text{ in } L^\infty((0,T);\mathcal{M}(\Omega)).$$

Measure-valued solutions

As a short-hand notation we will write

$$\overline{f}(t,x) = \langle \nu_{t,x}, f(\lambda) \rangle + m^f(t)(\mathrm{d}x).$$

Note that, the concentration measure $m^f \equiv 0$ if the sequence $\{f(z_j)\}$ is weakly relatively precompact in $L^1(\Omega)$. In our case, we shall use dummy variables
$\lambda = (s, v, F, G) \in [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ to express the ones in the equation. As an example

$$\begin{aligned}
\overline{\rho} &= \langle \nu, s \rangle \\
\overline{\rho u} &= \langle \nu, sv \rangle + m^{\rho u} \\
\overline{\nabla \Phi_{\overline{\rho}}} &= \langle \nu, F \rangle \\
\overline{\rho \nabla \rho} &= \langle \nu, sG \rangle + m^{\rho \nabla \rho}
\end{aligned}$$

and similarly for other terms.

Cahn-Hillard and Keller-Segel systems as high-friction limits of Euler-Korteweg and Euler-Poisson equations. D. Gallenmüller, P. Gwiazda, A. ŚG, J. Woźnicki, Calc. Var PDE. 2024

Theorem (Keller-Segel)

Let $r_0 \in C^2(\mathbb{T}^3)$ be initial data giving rise to a strong solution $(r, \Phi) \in C^2$ of Keller-Segel and for all $\varepsilon > 0$ let (ν_0^ε) be well-prepared measure-valued initial data in the sense that

$$\mathcal{E}_{\text{rel}}^\varepsilon(0) \to 0.$$

Then for all $\tau \in [0, T]$ and for any high friction sequence of dissipative measure-valued solutions (ν^ε) of Euler-Poisson with initial data $(\rho_0^\varepsilon, u_0^\varepsilon)$ it holds that

$$\mathcal{E}_{\text{rel}}^\varepsilon(\tau) + \frac{1}{2\varepsilon^2} \int_0^\tau \int_{\mathbb{T}^3} \overline{\rho^\varepsilon |u^\varepsilon - U^\varepsilon|^2} \mathrm{d}x\mathrm{d}t \to 0 \text{ as } \varepsilon \to 0.$$

Here, the relative entropy

$$\mathcal{E}_{\text{rel}}^{\varepsilon}(\tau) := \int_{\mathbb{T}^3} \frac{1}{2}\overline{\rho^{\varepsilon}|u^{\varepsilon} - U^{\varepsilon}|^2} + a\overline{h(\rho^{\varepsilon}|r)} + \frac{1}{2}\overline{|\nabla\Phi_{\rho^{\varepsilon}} - \nabla\Phi_r|^2}\mathrm{d}x,$$

is considered, where

$$U^{\varepsilon} := \varepsilon\nabla(ah'(r) - \Phi_r).$$

and

$$h(r) := \frac{a}{\gamma - 1}r^{\gamma}, \qquad h(\overline{\rho}|r) := h(\overline{\rho}) - h(r) - h'(r)(\overline{\rho} - r).$$

How does it work?

Let (r, Φ_r) be a strong solution to Keller-Segel equation, then (r, U, Φ_r), where

$$U = -\varepsilon\nabla\left(\frac{a\gamma}{\gamma-1}r^{\gamma-1} - b\Phi_r\right)$$

is a strong solution to the system

$$\partial_t r + \frac{1}{\varepsilon}\operatorname{div}(rU) = 0,$$
$$\partial_t(rU) + \frac{1}{\varepsilon}\operatorname{div}(rU\otimes U) + \frac{1}{\varepsilon^2}rU = -\frac{a}{\varepsilon}\nabla r^\gamma - \frac{b}{\varepsilon}\rho\nabla\Phi_\rho + e(r, U),$$
$$-\Delta\Phi_r = r - M_r,$$

for

$$e = e(r, U) := \partial_t(rU) + \frac{1}{\varepsilon}\operatorname{div}(rU\otimes U)$$

Relative entropy inequality

Now we need a tool in the form of the relative entropy inequality

$$\mathcal{E}^{\varepsilon}_{\mathrm{rel}}(\tau) + \frac{1}{2\varepsilon^2}\int_0^{\tau}\int_{\mathbb{T}^d} \overline{\rho^{\varepsilon}|u^{\varepsilon} - U^{\varepsilon}|^2}\,\mathrm{d}x\,\mathrm{d}t \leq \mathcal{E}^{\varepsilon}_{\mathrm{rel}}(0) + C\int_0^t \mathcal{E}^{\varepsilon}_{\mathrm{rel}}(t)\,\mathrm{d}t + O(\varepsilon^4)$$

for all $\tau \in [0, T]$, where C is some constant depending only on the strong solution (r, Φ_r).

- An analogue result holds for Euler-Korteweg and Cahn-Hillard.
- Existence of measure-valued (global in time) solutions is easy to get both for Euler-Korteweg and Euler-Poisson

Cahn-Hilliard as a limit system

- Recall that the proof relies on certain regularity of solutions of the limit system, which is not available in the case of the local Cahn-Hilliard equation.
- Therefore, we introduce an intermediate step and consider the nonlocal Cahn-Hilliard equation by introducing the parameter η.
- Since we know that the solutions to the nonlocal Cahn-Hilliard equation converge to the weak solutions of the local Cahn-Hilliard equation when $\eta \to 0$, it remains to prove that the nonlocal Euler-Korteweg system tends to the nonlocal Cahn-Hilliard equation when $\to 0$. Then, sending ε and η to 0 with the appropriate scaling, we prove the result.

The operator B_η is defined by

$$B_\eta[u](x) = \frac{1}{\varepsilon^2}(u(x) - \omega_\eta * u(x)) = \frac{1}{\eta^2}\int_\varepsilon \omega_\eta(y)(u(x) - u(x-y))dy,$$

and ω_η is the usual symmetric mollification kernel. Formally, it is clear that when $\eta \to 0$, $B_\eta[u] \approx -\Delta u$.

Regualrity for singular Cahn-Hillard

$$\partial_t \rho + \mathrm{div}(-a\nabla\rho^\gamma + \rho\nabla\Delta\rho) = 0,$$

Remark 1. Note that due to singularity of Cahn-Hillard only local in time result for regular solutions is expected.
Remark 2. As an approximation of this equation we can consider nonlocal version with operator B_η:

$$\partial_t \rho + \mathrm{div}(-a\nabla\rho^\gamma + \rho\nabla B_\eta\rho) = 0,$$

- *Degenerate Cahn-Hilliard equation: From nonlocal to local* C. Elbar, J. Skrzeczkowski, JDE (2023).
- *Degenerate Cahn-Hilliard systems: From nonlocal to local*, J. A. Carrillo, C. Elbar, J. Skrzeczkowski, arXiv:2303.11929 (2023)

Strategy

- We rigorously derive the nonlocal Cahn-Hilliard equation as a limit of the nonlocal Euler-Korteweg equation using the relative entropy method.
- We introduce the strategy to derive equations not enjoying classical solutions via relative entropy method by introducing the nonlocal effect in the fluid equation.

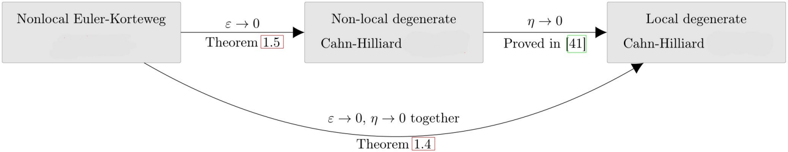

FIGURE 1. Relation between the three equations considered in this article.

From nonlocal Euler-Korteweg to local Cahn-Hilliard via the high-friction limit **C. Elbar, P. Gwiazda, J. Skrzeczkowski, A. Świerczewska-Gwiazda, arXiv:2305.01348 (2023)**

$$\partial_t \rho + \frac{1}{\varepsilon} \operatorname{div}(\rho u) = 0,$$
$$\partial_t(\rho u) + \frac{1}{\varepsilon} \operatorname{div}(\rho u \otimes u) + \frac{1}{\varepsilon^2} \rho u = -\frac{a}{\varepsilon} \nabla \rho^\gamma + \frac{1}{\varepsilon} \rho \nabla B_\eta \rho,$$

(Euler–Korteweg)

After ε goes to zero:

$$\partial_t \rho + \operatorname{div}(-a \nabla \rho^\gamma + \rho \nabla B_\eta \rho) = 0,$$

(nonlocal Cahn–Hillard)

For nonlocal Cahn-Hillard we can get global regular solutions.

Problem of the existence of classical solutions

- We proposed to introduce nonlocality in the equation and introduce an intermediate step in the convergence analysis.
- Advantage: the nonlocal Cahn-Hilliard equation is in fact a porous medium equation. In particular, it satisfies the maximum principle and so, if the initial condition is positive, the solution remains positive and one can prove the existence and uniqueness of a classical solution.
- the nonlocal Cahn-Hilliard equation converges to the local one so that at the end, the nonlocality can be removed.

A few remarks about Cahn-Hilliard equation

- The existence and uniqueness of solutions for the Cahn-Hilliard system
$$\begin{cases} \partial_t \rho &= \operatorname{div}\left(m(\rho)\nabla\mu\right), \\ \mu &= -D\Delta\rho + F'(\rho), \end{cases}$$
depends on the properties of the mobility term $m(\rho)$ and the potential $F(\rho)$
- The presence of *degeneration* on the mobility impedes the analysis. Elliot-Garcke 1996 - existence result of weak solutions in the case of degenerate mobility and Dai-Du 2016 for some improvements.
- The uniqueness and the existence of classical solutions are open questions for this type of mobility (and this is the reason we introduced nonlocal CH).
- The nonlocal CH equation is nowadays a topic of intense research activity (Carrillo, Trusardi, Skarpa, Skrzeczkowski, Gal, Giacomin, Lebowitz).

Thank you for your attention