On the Navier-Stokes-like system with the dynamic slip boundary condition

(Long-time behaviour, attractors)

Michael Zelina

Charles University, Faculty of Mathematics and Physics,
Department of Mathematical Analysis

zelina@karlin.mff.cuni.cz

MPDE 2024
Prague, Czech Republic,
September 23.

The Problem

- $\Omega \subset \mathbb{R}^d$ is (bounded) $\mathcal{C}^{0,1}$ or (infinite) channel domain
- Parameters α, β, and force $\boldsymbol{F} = (\boldsymbol{f}, \boldsymbol{h})$
- $\mathbb{S}$ and $\boldsymbol{s}$ are monotone functions with r, ρ-growth condition

$$\partial_t \boldsymbol{u} - \operatorname{div} \mathbb{S}(D\boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla \pi = \boldsymbol{f} \text{ in } (0, T) \times \Omega,$$
$$\left.\begin{aligned} \beta \partial_t \boldsymbol{u} + \alpha \boldsymbol{s}(\boldsymbol{u}) + [\mathbb{S}(D\boldsymbol{u})\vec{n}]_\tau &= \beta \boldsymbol{h} \\ \boldsymbol{u} \cdot \vec{n} &= 0 \end{aligned}\right\} \text{ on } (0, T) \times \Gamma, \qquad \text{(NS)}$$
$$\text{and } \operatorname{div} \boldsymbol{u} = 0\,, \boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0.$$

- Modelling molten polymers – S. G. Hatzikiriakos ’12
- Questions: What does $\partial_t \boldsymbol{u}$ on Γ? Effects of α and β?

General results in 3D

- General existence theory in bounded 3D domains
  A. Abbatiello, M. Bulíček, E. Maringová Kokavcová '21
  $r > \frac{6}{5}$, $\rho \geq 2$; (also for maximal monotone r, ρ-graphs)
- Attractor with finite fractal dimension exists if $r \geq \frac{11}{5}$, $r \geq \rho$
  with D. Pražák '24
  Fractional time regularity and method of trajectories
- Attractor exists also for ρ-graph...Dimension?
- More explicit estimate for $r > \frac{12}{5}$ (testing by $\partial_t \boldsymbol{u}$)
  D. Pražák, B. Priyasad '24
- Focus on 2D case

Regularity in 2D

Theorem 1

Let $\boldsymbol{u}_0 \in H$, $r = 2$. *[*$V = W^{1,2}_{div,\vec{n}}(\Omega)$, $H = L^2_{div,\vec{n}}(\Omega) \times L^2_{\vec{n}}(\Gamma)$*]*

(i) $\boldsymbol{F} \in L^2(0,T;V^*) \Rightarrow \exists!$ *weak solution to (NS), i.e.*

$$\boldsymbol{u} \in L^\infty(0,T;H) \cap L^2(0,T;V) \text{ and } \partial_t \boldsymbol{u} \in L^2(0,T;V^*).$$

(ii) *Moreover, let* $\partial_t \boldsymbol{F} \in L^2(0,T;V^*)$, $\boldsymbol{F}(0) \in H$, *and* $\mathbb{S}$, $\boldsymbol{s} \in \mathcal{C}^1$. *Then*

$$\partial_t \boldsymbol{u} \in L^\infty_{loc}(0,T;H) \cap L^2_{loc}(0,T;V) \Rightarrow \boldsymbol{u} \in L^\infty_{loc}(0,T;V).$$

Theorem 2

Let $\boldsymbol{u}_0 \in H$ *and* $\boldsymbol{F} \in V^*$. *Suppose* $\Omega \in \mathcal{C}^{1,1}$, $\mathbb{S}$, $\boldsymbol{s} \in \mathcal{C}^2$, *and* $\boldsymbol{F} \in (L^p(\Omega) \times W^{1-\frac{1}{p},p}(\Gamma))$ *for some* $p \geq 2$.
Then the unique weak solution satisfies (for certain $q \geq 2$*)*

$$\boldsymbol{u} \in L^\infty_{loc}(0,T;W^{2,q}(\Omega)) \text{ and } \pi \in L^\infty_{loc}(0,T;W^{1,q}(\Omega)).$$

Attractor dimension in 2D

- Attractor dimension using Lyapunov exponents

Theorem 3
Let $\boldsymbol{F} \in H$, and $\Omega \in \mathcal{C}^{0,1}$ (with $\ell = \mathrm{diam}\,\Omega$) or $\Omega = \mathbb{R} \times (0, L)$. Then

$$\begin{aligned} \dim_H^f \mathcal{A} &\lesssim \frac{\max\{1, \beta/\ell\}}{\min^{3/2}\{1, \alpha\ell\}} \cdot \frac{\ell^2}{\nu^2} ||\boldsymbol{F}||_H \,, \\ &\lesssim \left(\frac{32 L^2}{\pi^2} + \beta \min\{1/\alpha, L\} \right)^2 \cdot \left(\frac{||\boldsymbol{F}||_H}{\nu^2} \right)^2 . \end{aligned}$$

- $\alpha \to +\infty$, $\beta \to 0_+$, and $\alpha \to 0_+$
- with D. Pražák '24 and M. Zelina '25 (preprint)

Future goals

- Further improving upper bounds ($\alpha \to 0_+$ for bounded Ω?)
- Lower bounds for $\dim_H^f \mathcal{A}$
- Half-space (i.e. $L = +\infty$)
- Regularity for more general $\mathbb{S}$
- "$\boldsymbol{s}(\boldsymbol{u}) = \operatorname{sgn} \boldsymbol{u}$"...jumps
- Passing with $\alpha \to +\infty$ or $\beta \to 0_+$