(Non-)existence of Phase Transition Solutions for a Quasilinear Elliptic Problem (via Optimal Regularity and Pohozhaev's Identity)

A stationary Cahn-Hilliard model with the p-Laplacian: radial solutions and pattern formation

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Time-dependent Cahn-Hilliard equation:

(1) $$u_t = \Delta \left[-\varepsilon^p \, \nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right) + W'(u) \right]$$
$$\text{for } \; x \in \Omega \; \text{ and } \; t > 0 \,,$$

subject to the Neumann (i.e., no-flux) boundary conditions

(2) $$|\nabla u|^{p-2} (\boldsymbol{\nu} \cdot \nabla) u = 0 \quad \text{ and }$$
$$(\boldsymbol{\nu} \cdot \nabla) \left[-\varepsilon^p \, \nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right) + W'(u) \right] = 0$$
$$\text{at } \; x \in \partial\Omega \; \text{ for } \; t > 0 \,,$$

where $1 < p < \infty$, $\varepsilon > 0$, and $W : \mathbb{R} \to \mathbb{R}$ is a given potential function of class C^1 whose first derivative might be only Hölder-continuous. Semi-linear case $p = 2$ is well-known; see, e.g., R. Temam (monograph by Springer-Verlag).

We abbreviate by $\quad \Delta_p u \stackrel{\text{def}}{=} \nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right)$
the well-known p-Laplace operator; of course, $\Delta_2 \equiv \Delta$ is the (linear) Laplace operator.

Material occupies a bounded domain $\Omega \subset \mathbb{R}^N$ with a sufficiently smooth boundary $\partial\Omega$.

If W is of class C^2 then
the boundary conditions (2) are equivalent with
the Navier boundary conditions

$$(\boldsymbol{\nu} \cdot \nabla)u = (\boldsymbol{\nu} \cdot \nabla)(\Delta_p u) = 0 \quad \text{at } x \in \partial\Omega \ \text{ for } \ t > 0\,. \tag{3}$$

Classical choice of W: double-well potential
$W(s) = (1 - s^2)^2$ for $s \in \mathbb{R}$
which attains global minimum at two points, $s_1 = -1$ and $s_2 = 1$ (Cahn and Hilliard).
These points of minimum are nondegenerate,
with $W'(\pm 1) = 0$ and $W''(\pm 1) = 8 > 0$.

Entirely different behavior of
the stationary solutions satisfying

$$-\,\varepsilon^p\,\Delta_p u + W'(u) = 0 \qquad \text{for } \ x \in \Omega\,; \tag{4}$$

$$(\boldsymbol{\nu} \cdot \nabla)u = 0 \qquad \text{at } \ x \in \partial\Omega\,, \tag{5}$$

for classical linear diffusion ($p = 2$) and
degenerate nonlinear diffusion ($p > 2$).

Degenerate nonlinear diffusion ($p > 2$) exhibits a greater variety of stationary solutions.

One can observe the same phenomenon for classical linear diffusion ($p = 2$) if the potential W is modified to $W(s) = |1 - s^2|^\alpha$ for $s \in \mathbb{R}$, where α is a constant, $1 < \alpha < 2$.

We focus on problem (4), (5) with arbitrary $p, \alpha > 1$. More generally, we consider the potential

$$W(s) = \left|1 - |s|^\beta\right|^\alpha \qquad \text{for} \quad s \in \mathbb{R}\,,$$

where $p, \alpha, \beta > 1$ are constants.

Asymptotic behavior:

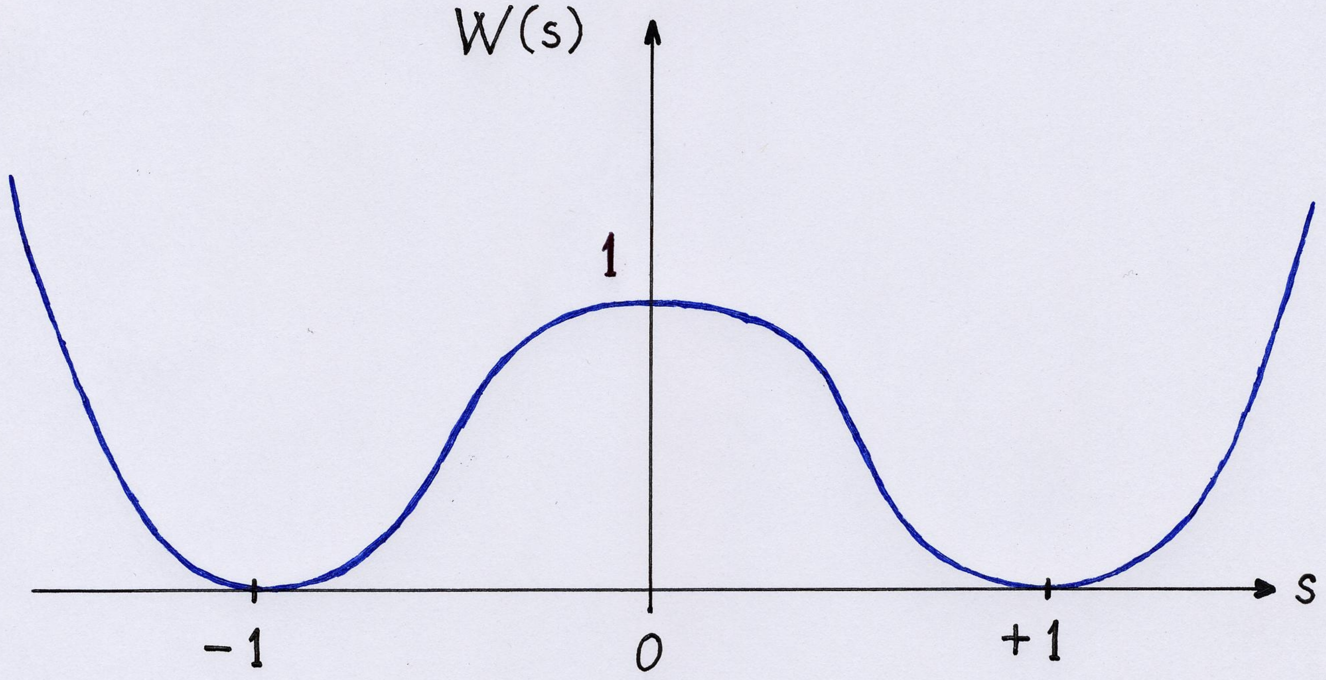
$$W(s) = \beta^\alpha\, |1 - |s||^\alpha + \mathcal{O}\left(|1 - |s||^{\alpha+1}\right) \qquad \text{for} \quad s \to \pm 1\,,$$

$$W(s) = 1 - \alpha\, |s|^\beta + \mathcal{O}\left(|s|^{2\beta}\right) \qquad \text{for} \quad s \to 0\,.$$

The following constant is important for $N \geq 2$,

$$p_{N,\beta} \stackrel{\text{def}}{=} \frac{\beta N}{N - 1 + \beta} \geq 1\,.$$

One has $p_{1,\beta} = 1$ and $p_{N,\beta} > 1$ for $N \geq 2$.



$$W(s) = \big|\, 1 - |s|^{\beta} \big|^{\alpha} \qquad \text{for} \quad s \in \mathbb{R},$$

$\alpha, \beta > 1$ are constants.

- $$W(s) = \beta^{\alpha} \cdot \big|1 - |s|\big|^{\alpha} + \mathcal{O}\left(\big|1 - |s|\big|^{\alpha+1}\right) \quad \text{as } |s| \to 1 \,;$$

- $$W(s) = 1 - \alpha \cdot |s|^{\beta} + \mathcal{O}\left(|s|^{2\beta}\right) \quad \text{as } |s| \to 0 .$$

$$p_{N,\beta} \overset{def}{=} \frac{\beta N}{N - 1 + \beta} \quad \Longrightarrow \quad p_{1,\beta} = 1 < p$$

Radially symmetric solutions in a ball

Boundary value problem (4), (5) reduces to

$$-\,\varepsilon^p\, r^{-(N-1)} \left(r^{N-1}\, |u'|^{p-2} u'\right)' + W'(u) = 0, \quad 0 < r < R; \tag{6}$$

$$u'(0) = 0, \quad u'(R) = 0. \tag{7}$$

This problem applies to any choice $N \geq 1$.

$$\varepsilon^p\, (|u'|^{p-2} u')' + \varepsilon^p\, \frac{N-1}{r}\, |u'|^{p-2} u' - W'(u) = 0, \quad 0 < r < \infty; \tag{8}$$

$$u(0) = -\theta, \quad u'(0) = 0, \tag{9}$$

where $u_0 = -\theta \in [-1, 1]$ is given.

Shooting method – three (3) factors:

- reaction (independent from N)
- diffusion (for every $N \geq 1$)
- damping (only for $N \geq 2$)

Case $N = 1$.

(Pavel Drábek, U.W.B., Plzeň, Czech Rep.,
Raúl F. Manásevich, Univ. de Chile, Santiago)

Problem (4), (5) is the boundary value problem for all *stationary solutions* of the so-called *bi-stable equation*

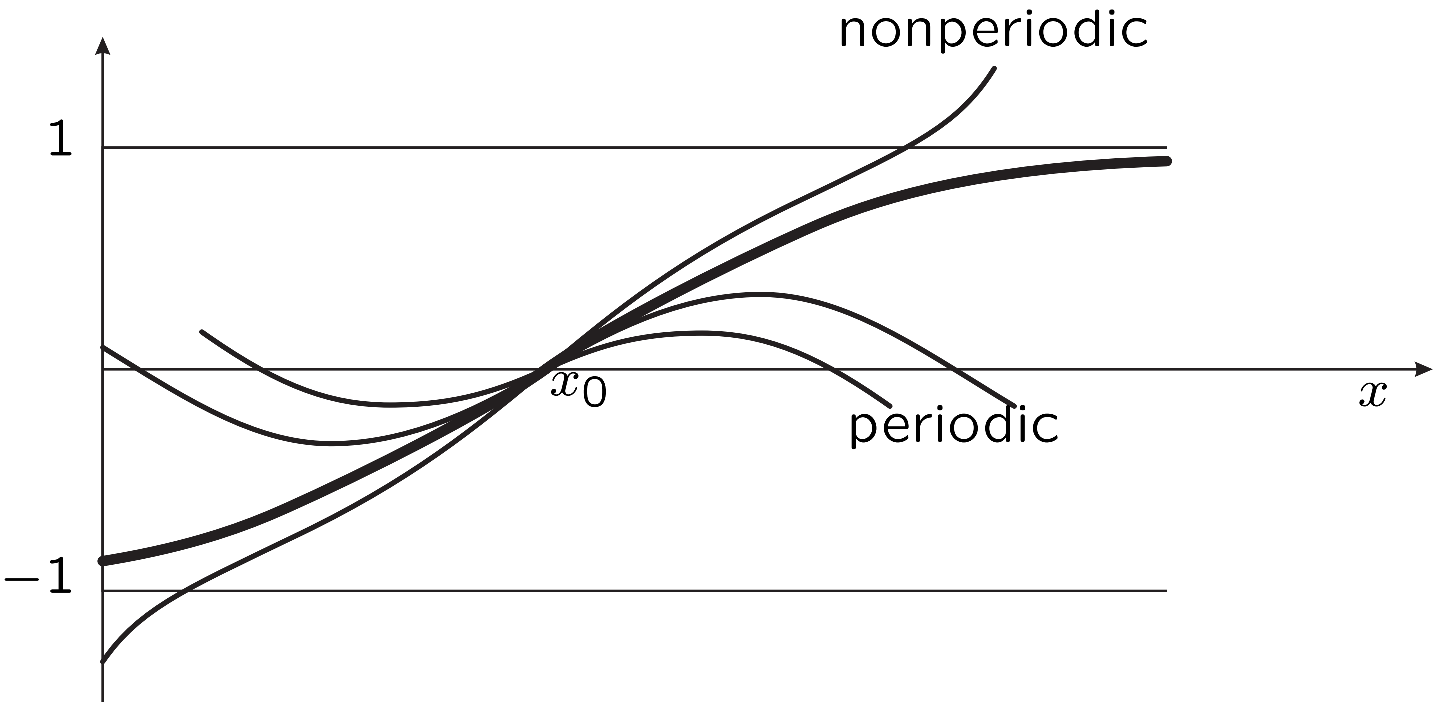
$$u_t = \varepsilon^p \left(|u_x|^{p-2} u_x \right)_x - W'(u) \quad \text{for } 0 < x < 1 \text{ and } t > 0, \tag{10}$$

subject to the boundary conditions

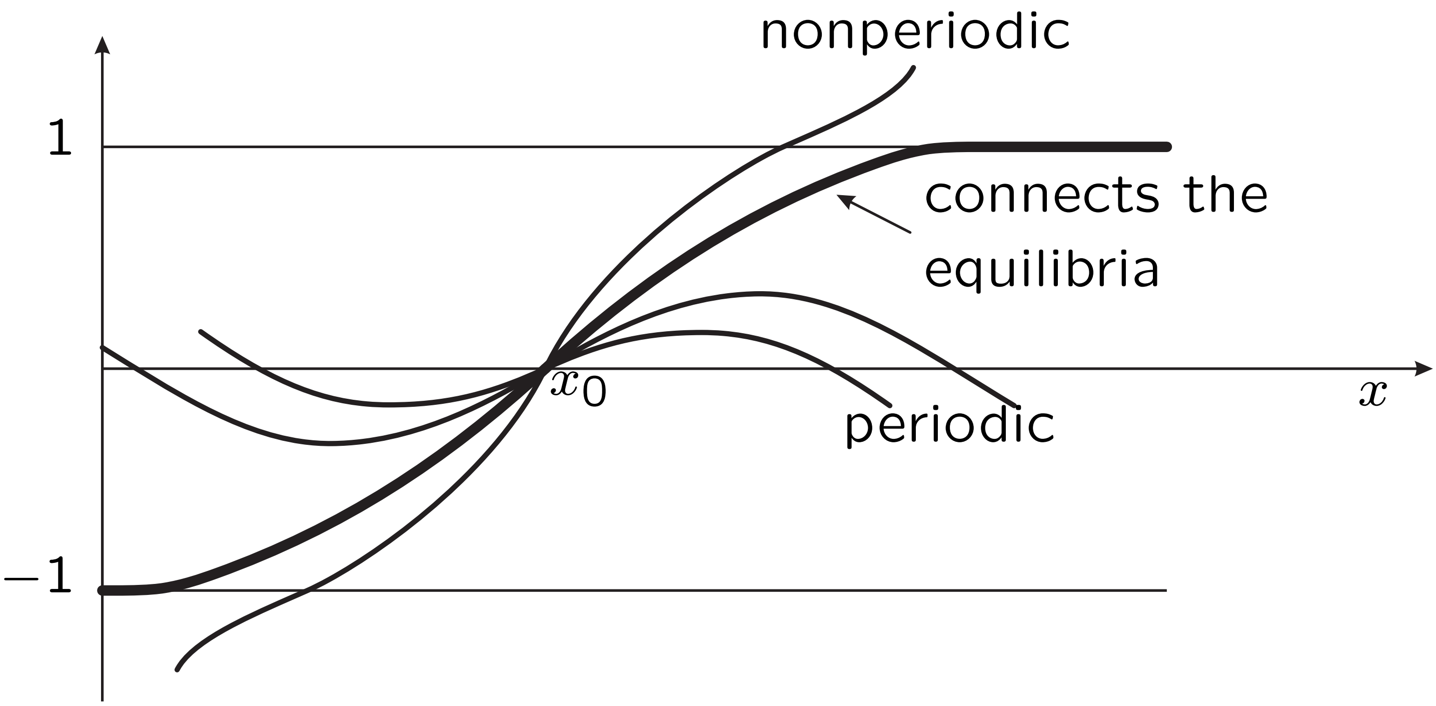
$$u_x = 0 \quad \text{at } x = 0,\, 1, \quad \text{for } t > 0. \tag{11}$$

Equations (10), (11) describe the **gradient flow** for the (total free energy) functional

$$\mathcal{J}_\varepsilon(u) \stackrel{\text{def}}{=} \int_0^1 \left(\frac{\varepsilon^p}{p} |u_x|^p + W(u) \right) \mathrm{d}x, \quad u \in W^{1,p}(0,1). \tag{12}$$



$1 < p \leq \alpha < \infty$, $W(s) = |1 - s^2|^\alpha$.



$1 < \alpha < p < \infty$, $W(s) = |1 - s^2|^\alpha$.

For $1 < \alpha < p < \infty$ we have loss of uniqueness in the initial value problem for the first integral of eq. (8),

$$\frac{p-1}{p}\,\varepsilon^p\,|u_x(x)|^p - W(u(x)) = \text{const}, \qquad 0 \le x \le 1. \tag{13}$$

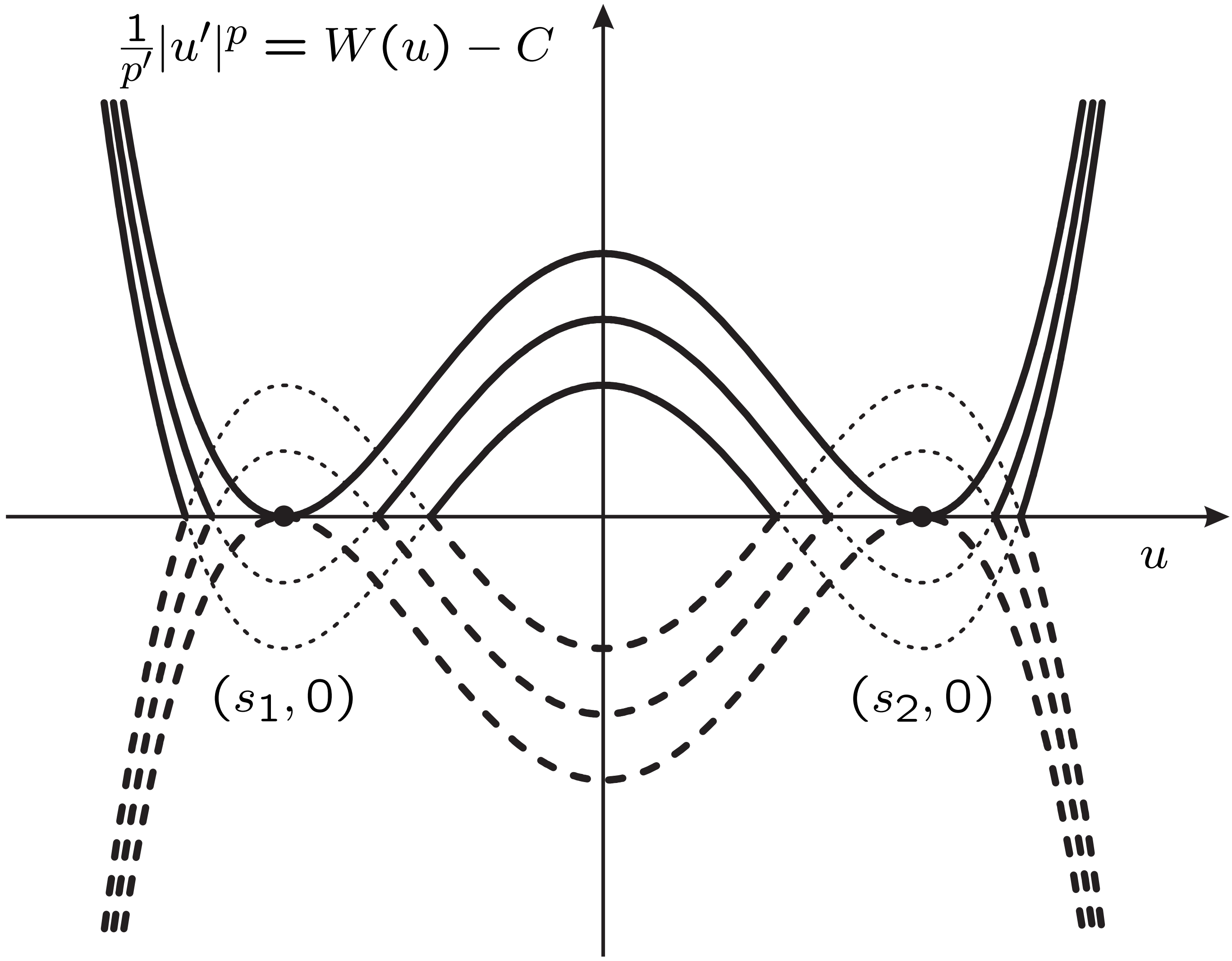
Denote $p' = p/(p-1)$. Then eq. (13) reads (we look for a local solution u)

$$u' = \operatorname{sgn} u_0' \cdot \left[p'(W(u) - C)\right]^{1/p}; \tag{14}$$

$$u(x_0) = u_0\,. \tag{15}$$

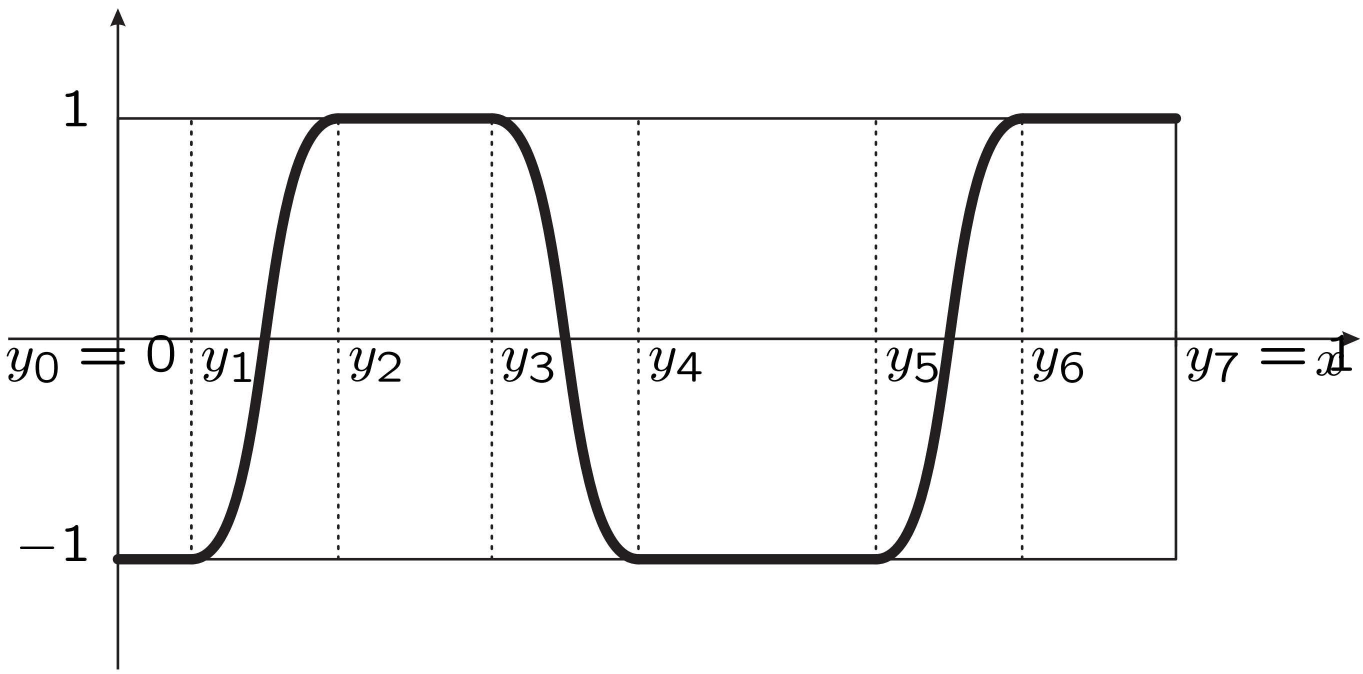
Here, we take $u_0' = u'(x_0)$ if $u'(x_0) \neq 0$ (hence, $W(u_0) - C > 0$); otherwise $C = W(u_0)$ and we may take any number $u_0' \in \mathbb{R}$.

The right-hand side is locally Lipschitz if $u'(x_0) \neq 0$; local existence and uniqueness.



Shifts of the potential W by a constant C.

A stationary solution for $1 < \alpha < p < \infty$ which gives a nonperiodic pattern formation:



Function u_ε for $m = 3$.

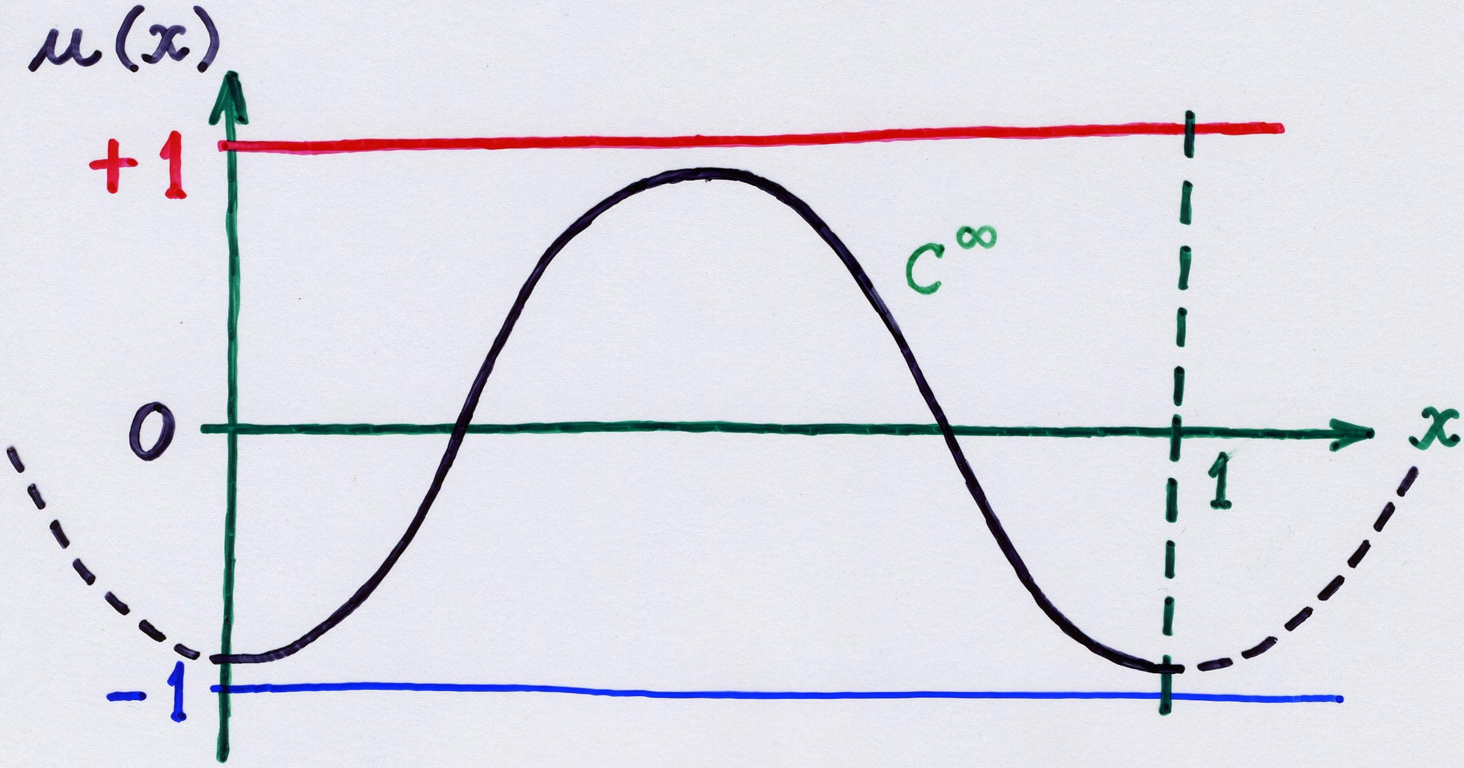
A partition of the interval $[0, 1]$ with the points

$$
\begin{aligned}
&0 = y_0 \le y_1 < y_2 \le y_3 < \cdots < \\
&y_{2k} \le y_{2k+1} < \cdots < y_{2m} \le y_{2m+1} = 1,
\end{aligned}
\tag{16}
$$

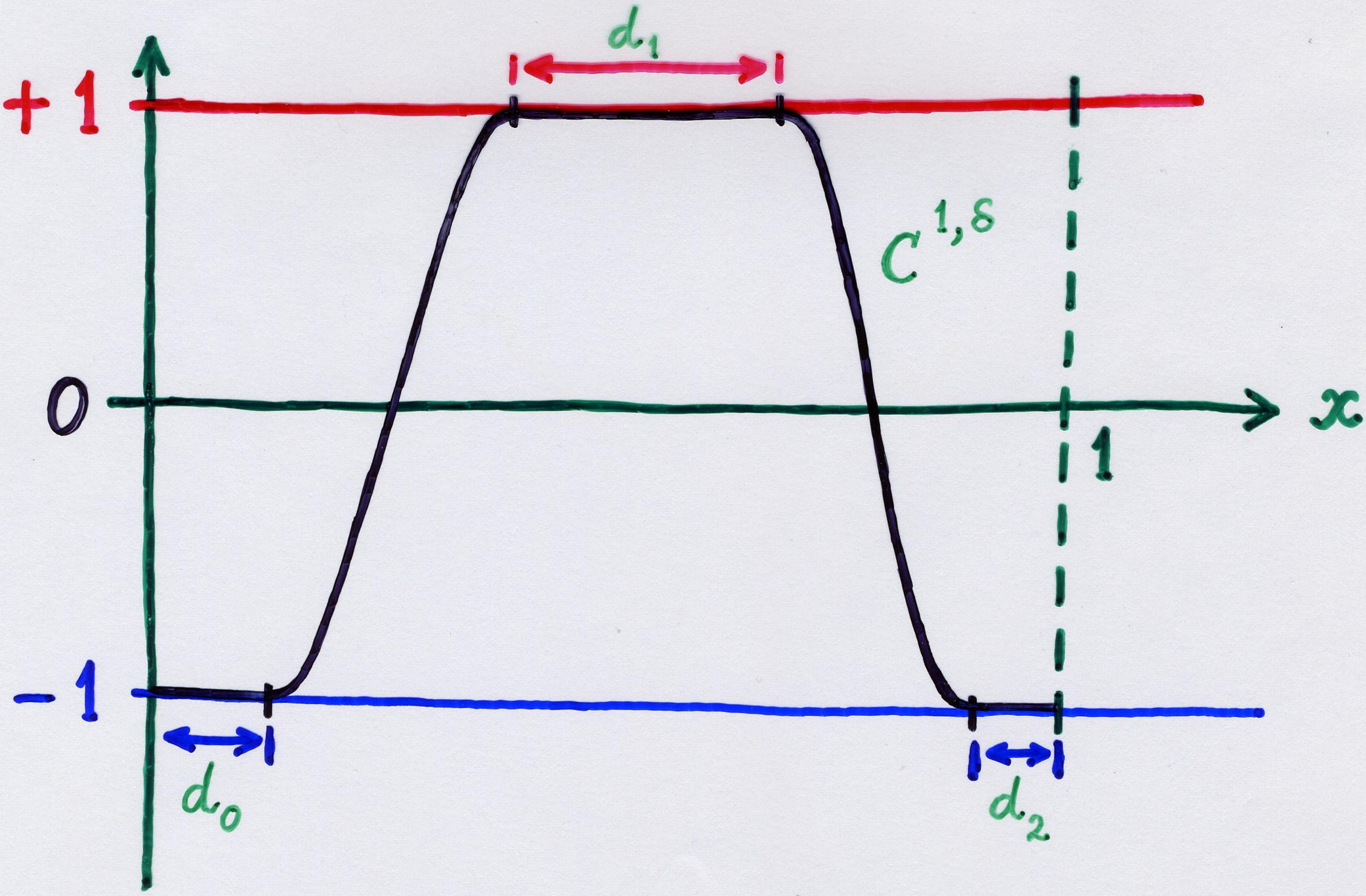
where $k = 0, 1, 2, \ldots, m$ ($m \ge 0$ − an integer),

$$
\begin{aligned}
&y_{2k} - y_{2k-1} = 2\vartheta_\varepsilon = 2\varepsilon\vartheta_1 \quad (> 0)\,, \\
&y_{2k+1} - y_{2k} = d_k \ge 0\,.
\end{aligned}
\tag{17}
$$

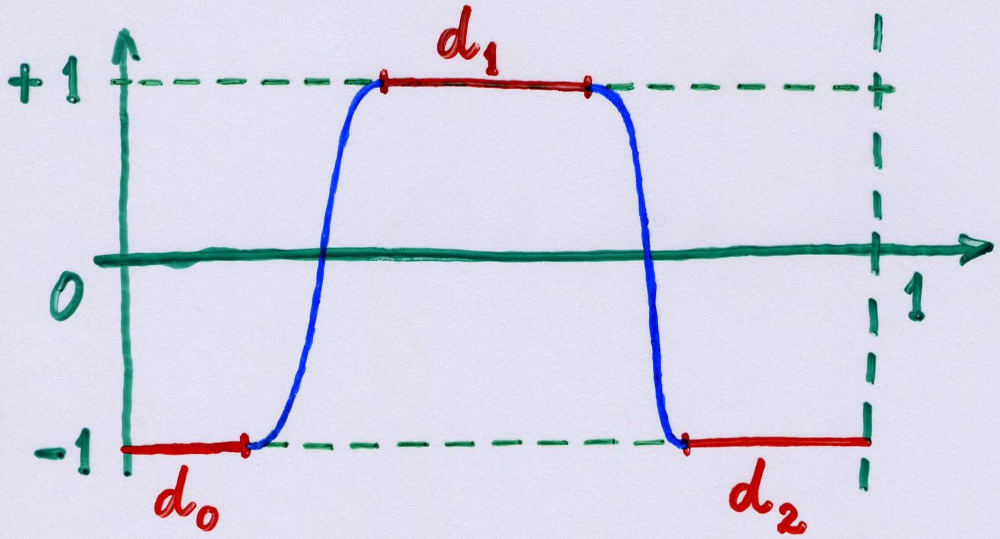
1 – D case



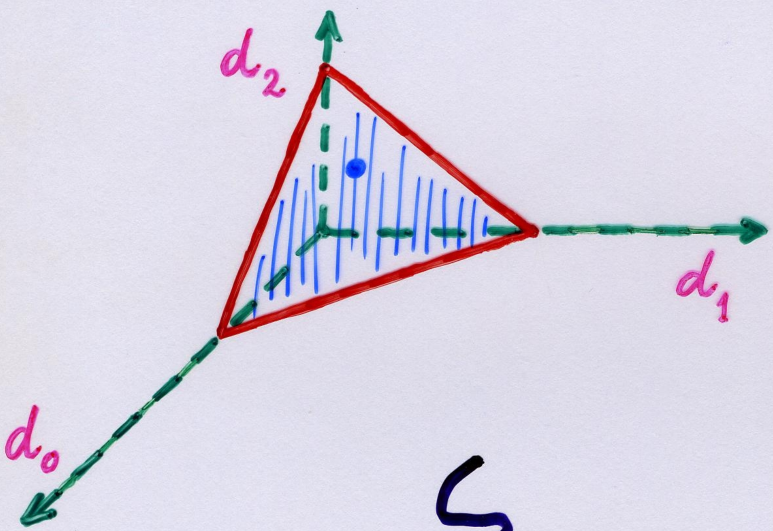
$p = \alpha = 2$

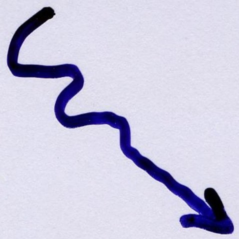


$p > \alpha = 2$ / $p = 2 > \alpha > 1$

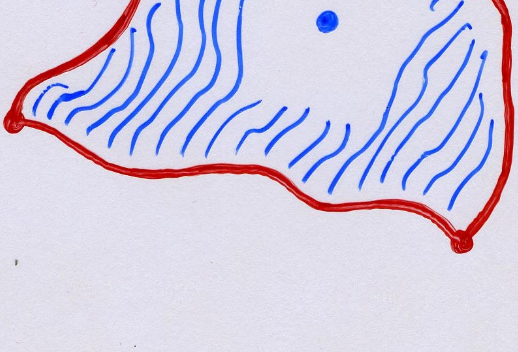


$$d_0 + d_1 + d_2 = \text{const}(\varepsilon) > 0$$





$$W^{1,p} \hookrightarrow L^2$$



Case $N \geqq 2$.

(Electr. J.D.E., Conf. **17** (2009), 227–254)

The first integral of eq. (8) gets a damping term $Z(r)$,

$$\frac{\varepsilon^p}{p'} |u'(r)|^p - W(u(r)) + \varepsilon^p Z(r) = 0, \quad 0 \leq r < \infty, \tag{18}$$

where the damping term $Z(r)$ satisfies

$$Z'(r) = \frac{N-1}{r} |u'(r)|^p \geq 0 \qquad \text{for } r > 0. \tag{19}$$

Rescaling: We replace r by $\varepsilon^{-1} r$ to get $\varepsilon = 1$.

If $u(0) = s_0 = \pm 1$ (a local minimizer for W) and $u'(0) = 0$, then the functions

$$r \mapsto r^{-1} |u'(r)|^{p-1} \quad \text{and}$$
$$r \mapsto r^{-p'} (Z(r) - Z(0)) : (0, \delta) \to \mathbb{R}_+$$

are monotone increasing, for some $\delta > 0$ small. (W is convex in an open interval containing s_0.)

From these facts we derive the inequalities

$$(20)\qquad \begin{aligned} (0 \le\ ) \quad & \tfrac{1}{N}\left[W(u(r)) - W(u(0))\right] \\ & \le W(u(r)) - Z(r) \\ & \le W(u(r)) - W(u(0)) \end{aligned}$$

for all $r \in [0, \delta)$, where
$Z(0) = W(u(0)) = W(s_0)$.
Applying (20) to (18), we arrive at

$$(21)\qquad \begin{aligned} & \frac{p'}{N}\left[W(u(r)) - W(u(0))\right] \\ & \le |u'(r)|^p \le p'\left[W(u(r)) - W(u(0))\right] \end{aligned}$$

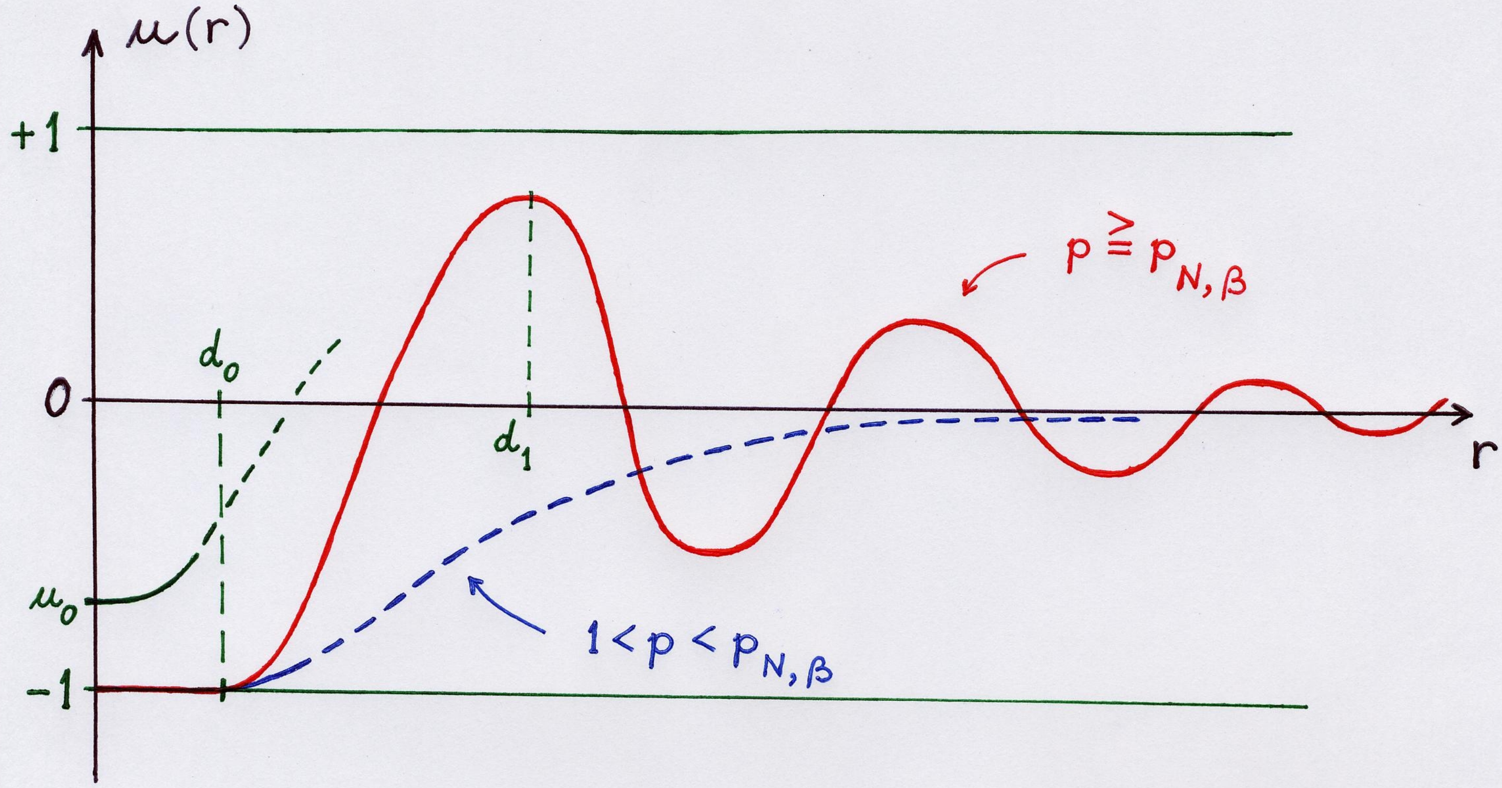
for all $r \in [0, \delta)$.

This means the *uniqueness* or *nonuniqueness* of a local solution u to equation (8) with
the initial conditions $u(0) = s_0$
(where s_0 is a local minimizer for W)
and $u'(0) = 0$ at $r = 0$, depending on

$$(22)\qquad \int_{s_0}^{s_0+\zeta} |W(s) - W(s_0)|^{-1/p}\, \mathrm{d}s$$

being *infinite* (forcing uniqueness)
or *finite* (forcing nonuniqueness), respectively:
$p \le \alpha$ (infinite) or $p > \alpha$ (finite).

$N \geqq 2$: $u(x) \equiv u(r)$ with $r = |x|$



$d_0 = 0$ if $1 < p \leqq \alpha$ (strong diffusion)

$d_0 \geqq 0$ if $p > \alpha$ (weak diffusion)

$d_0 = 0$ if $|u(0)| < 1$, $u(0) \neq 0$

3 Factors: reaction
diffusion ($N \geqq 1$)
damping ($N \geqq 2$)

$$\frac{1}{p_{N,\beta}} = \frac{1}{N} + \frac{1}{\beta} - \frac{1}{\beta N} \leqq 1$$

Isolated pure state regions ($N \geq 2$)

(Yavdat Sh. Il'yasov, Ufa, Russia)
(Russian Academy of Sciences)

Problem. $\Omega, \Omega_1 \subset \mathbb{R}^N$ bounded domains with C^2-boundaries, $\overline{\Omega_1} \subset \Omega$ and the open set $\Omega_1' = \Omega \setminus \overline{\Omega_1}$ is connected. $u : \overline{\Omega_1'} \to \mathbb{R}$ is called a *phase transition solution* of

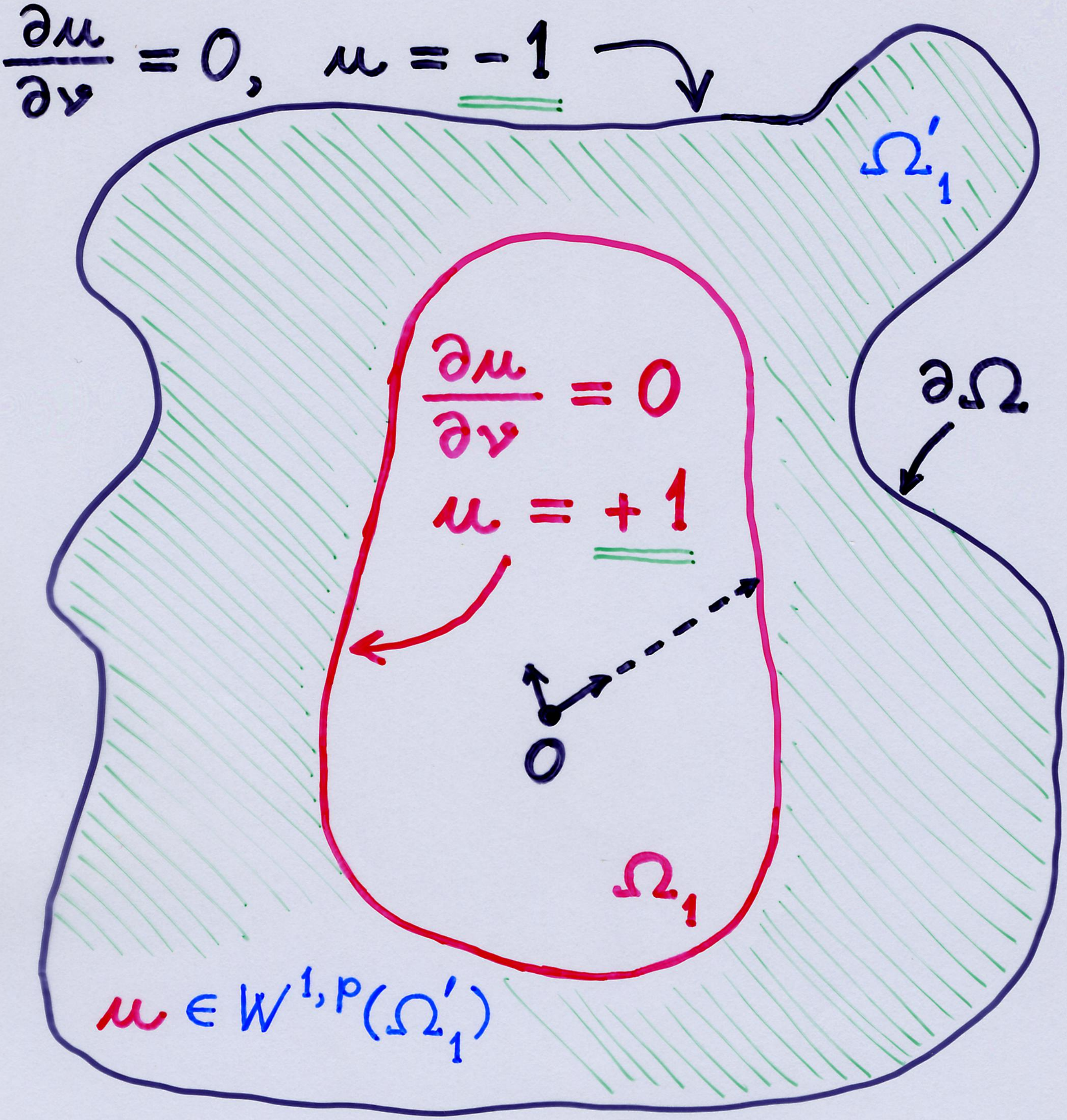
(23) $$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + W'(u) = 0 \quad \text{in } \Omega_1'$$

if $u \in W^{1,p}(\Omega_1')$ verifies eq. (23) above (in the weak sense) with the Neumann boundary conditions

(24) $$\partial u / \partial \nu = 0 \quad \text{on } \partial\Omega_1' = \partial\Omega \cup \partial\Omega_1$$

and the "phase transition" property

(25) $$u = -1 \text{ on } \partial\Omega \text{ and } u = 1 \text{ on } \partial\Omega_1 .$$



$$\begin{cases} -\Delta_p u + W'(u) = 0 & \text{in } \Omega_1'; \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_1' = \partial\Omega \cup \partial\Omega_1. \end{cases}$$

Theorem.
(Nonexistence of a phase transition solution)
Ω and Ω_1 as above and $1 < p \le N$. Furthermore, let Ω_1 be star-shaped with respect to the origin $\mathbf{0} \in \mathbb{R}^N$. Then any weak solution $u \in W^{1,p}(\Omega_1')$ of eq. (23) with the Neumann boundary conditions (24), such that $|u| = 1$ on $\partial\Omega$, must be a constant $= \pm 1$.

Proof is based on

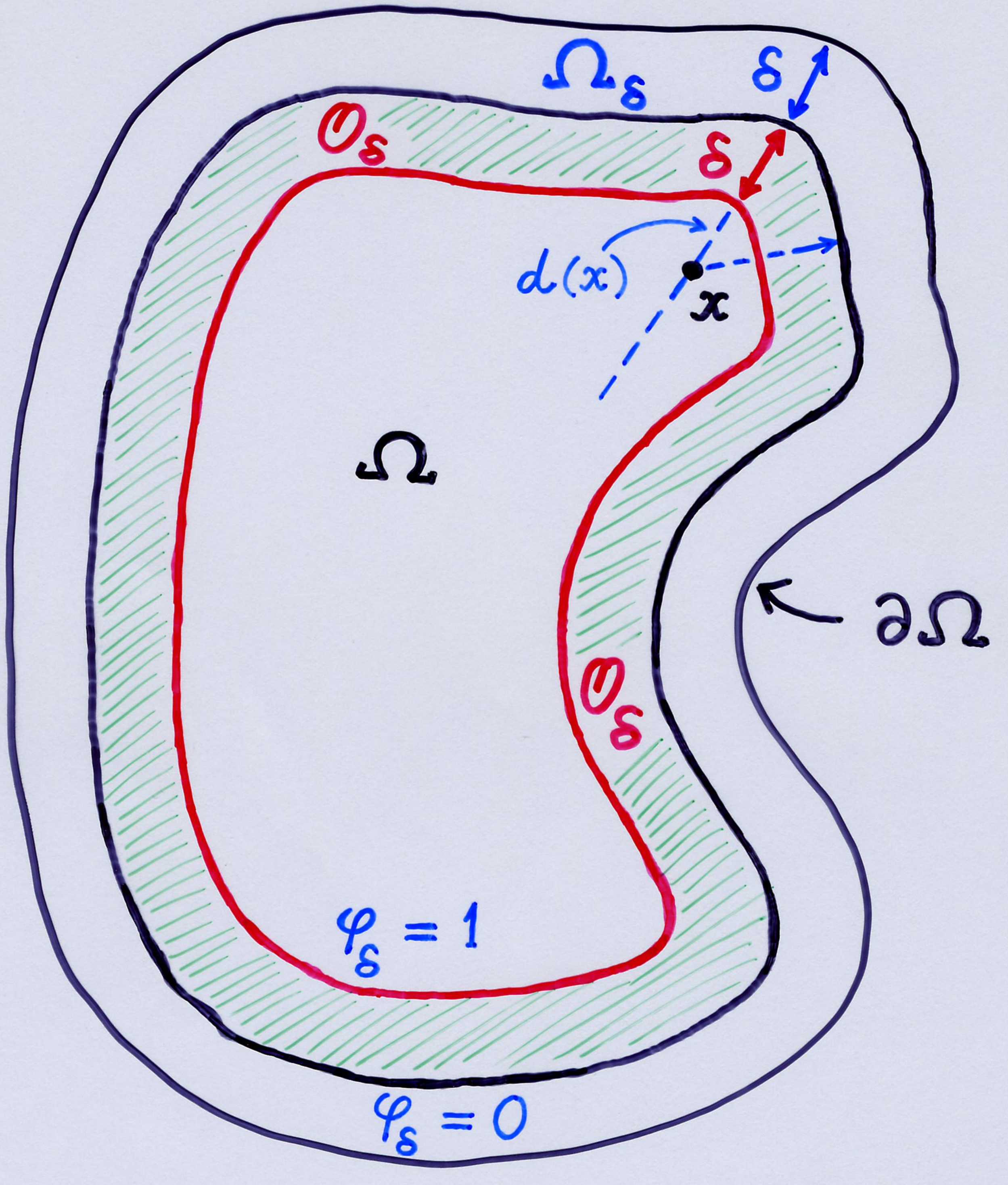
Proposition 1. (Pohozhaev's identity)
Let $1 < p < \infty$. Assume $f \in L^1(\Omega)$ possesses derivatives $\partial f / \partial x_i \in L^1_{\mathrm{loc}}(\Omega)$; $i = 1, 2, \dots, N$. Let $u \in C^1(\overline{\Omega})$ satisfy

(26) $$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) = f(x) \quad \text{in } \Omega$$

in the sense of distributions in Ω and $|\nabla u|^q \in W^{1,1}_{\mathrm{loc}}(\Omega)$ for some $q \in (1, p)$. Then

$$\frac{N-p}{p} \int_\Omega |\nabla u|^p \, \mathrm{d}x + \int_\Omega f(x)\,(x \cdot \nabla u)\, \mathrm{d}x$$
$$= \frac{1}{p} \int_{\partial\Omega} |\nabla u|^p \,(x \cdot \nu(x))\, \mathrm{d}\sigma(x)$$
$$- \int_{\partial\Omega} |\nabla u|^{p-2} \,(x \cdot \nabla u)(\nu(x) \cdot \nabla u)\, \mathrm{d}\sigma(x)\,.$$

$\delta > 0$ small, $1 < p < 2$



$$\int_{\mathcal{O}_\delta} |\nabla u|^{-(2-p)} \, dx < \infty$$

Proposition 2.
(Optimal regularity in weighted $W^{2,2}_{\text{loc}}$)
Assume $1 < p < \infty$; *if* $p < 2$ *then assume also*

Hypothesis. $\nabla u \neq \mathbf{0}$ *a.e. in* $\mathcal{O}_\delta$, *for* $\delta > 0$ *small enough, where*

$$\mathcal{O}_\delta = \{x \in \Omega'_\delta : d_\delta(x) < \delta\},$$
$$\Omega'_\delta = \Omega \setminus \overline{\Omega_\delta}, \quad \Omega_\delta = \{x \in \Omega : d(x) < \delta\},$$
$$d_\delta(x) = \begin{cases} 0 & \text{if } x \in \Omega_\delta; \\ \text{dist}(x, \Omega_\delta) & \text{if } x \in \Omega \setminus \Omega_\delta. \end{cases}$$

and

(27)
$$\int_{\mathcal{O}_\delta} |\nabla u|^{-(2-p)}\, \mathrm{d}x < \infty\,.$$

Let us construct $\varphi_\delta : \overline{\Omega} \to [0,1]$ *as follows:*
(28)
$$\varphi_\delta(x) = \begin{cases} (\delta^{-1} d_\delta(x))^2 & \text{if } x \in \overline{\Omega_\delta} \cup \mathcal{O}_\delta; \\ 1 & \text{if } x \in \Omega \setminus (\overline{\Omega_\delta} \cup \mathcal{O}_\delta). \end{cases}$$

(a) *Then the vector field* $|\nabla u|^{(p/2)-1}\nabla u$ *is in the local Sobolev space* $W^{1,2}_{\text{loc}}(\Omega'_\delta)$ *and*

$$\int_{\Omega'_\delta} \left|\nabla\left(|\nabla u|^{(p/2)-1}\nabla u\right)\right|^2 \varphi_\delta\, \mathrm{d}x < \infty\,.$$

(b) *Moreover,* $u \in W^{2,2}_{\text{loc}}(U)$ *where*

$$U = \Omega'_\delta \cap \{x \in \Omega : \nabla u(x) \neq \mathbf{0}\}$$

and one has

$$\int_U |\nabla u|^{p-2} \, |\nabla(\nabla u)|^2 \, \varphi_\delta \, \mathrm{d}x < \infty \, .$$

(c) *Finally, if* $p < 2$ *then* $u \in W^{2,2}_{\text{loc}}(\Omega'_\delta)$ *and* $\nabla(\nabla u)(x) = \mathbf{0} \in \mathbb{R}^{N\times N}$ *holds for almost every*

$$x \in U' = \Omega'_\delta \cap \{x \in \Omega : \nabla u(x) = \mathbf{0}\} \, .$$

Consequently, we may write

$$\int_{\Omega'_\delta} |\nabla u|^{p-2} \, |\nabla(\nabla u)|^2 \, \varphi_\delta \, \mathrm{d}x < \infty \, .$$

Remark 3. (The p-Laplacian)
For $1 < p < \infty$ *we have* $\Delta_p u$

$$= |\nabla u|^{p-2} \left(\Delta u + \frac{p-2}{|\nabla u|^2} \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \, \partial x_j} \right)$$

$$= |\nabla u|^{p-2} \sum_{i,j=1}^{N} \left(\delta_{ij} + \frac{p-2}{|\nabla u|^2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) \frac{\partial^2 u}{\partial x_i \, \partial x_j}$$

with $\left(\delta_{ij} + \frac{p-2}{|\nabla u|^2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right)_{i,j=1}^{N}$ *uniformly elliptic.*

Notice that the entries $A_{ij} = \partial a_i / \partial \eta_j$ of the Jacobian matrix $\mathbf{A} = (A_{ij})_{i,j=1}^{N}$ of the mapping $\boldsymbol{\eta} \mapsto \mathbf{a}(\boldsymbol{\eta}) \stackrel{\text{def}}{=} |\boldsymbol{\eta}|^{p-2}\boldsymbol{\eta} : \mathbb{R}^N \to \mathbb{R}^N$,

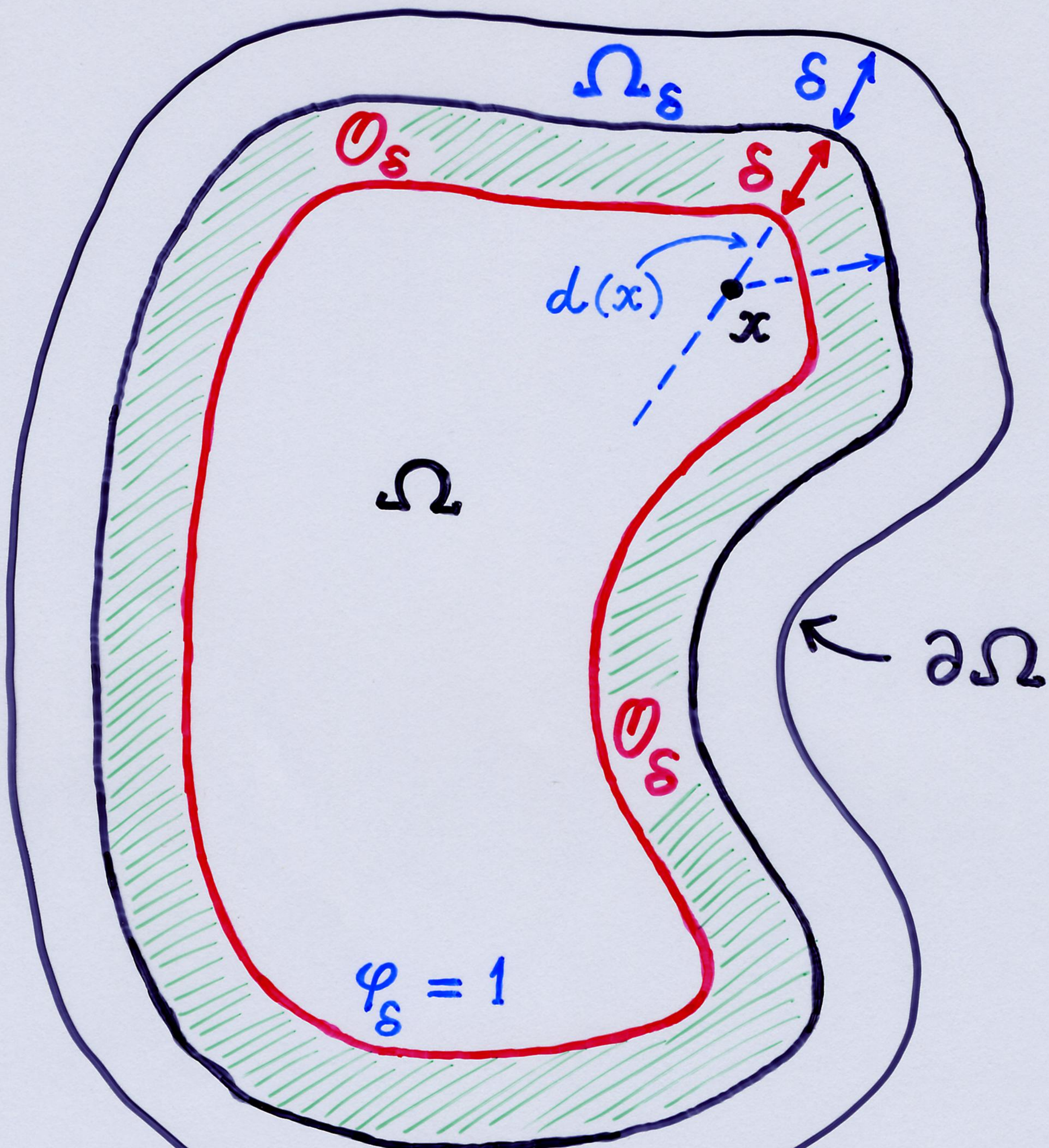
(29)
$$\mathbf{A}(\boldsymbol{\eta}) = |\boldsymbol{\eta}|^{p-2} \left(\mathbf{I} + (p-2) \frac{\boldsymbol{\eta} \otimes \boldsymbol{\eta}}{|\boldsymbol{\eta}|^2} \right) , \quad \boldsymbol{\eta} \in \mathbb{R}^N \setminus \{\mathbf{0}\} ,$$

satisfy the ellipticity and growth inequalities, with some constants $0 < \gamma \leq \Gamma < \infty$,

$$\begin{aligned} \gamma\, |\boldsymbol{\eta}|^{p-2}\, |\boldsymbol{\xi}|^2 \leq \langle \mathbf{A}(\boldsymbol{\eta})\boldsymbol{\xi},\, \boldsymbol{\xi} \rangle = \sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \eta_j}(\boldsymbol{\eta}) \cdot \xi_i \xi_j \\ \leq \Gamma\, |\boldsymbol{\eta}|^{p-2}\, |\boldsymbol{\xi}|^2 \end{aligned} \tag{30}$$

for all $\boldsymbol{\eta} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ and for all $\boldsymbol{\xi} \in \mathbb{R}^N$.

$\delta > 0$ small, $1 < p < 2$



$$\int_{\mathcal{O}_\delta} |\nabla u|^{-(2-p)}\, dx < \infty$$

Proof of **Proposition 2.**
Optimal regularity $|\nabla u|^{(p/2)-1}\nabla u \in W^{1,2}_{\mathrm{loc}}(\Omega'_\delta)$

Take $h \in \mathbb{R}^N$ with $0 < |h| < \delta$.
Let $\varphi \in C^1_0(\Omega)$, $\varphi \geq 0$, be supported in

$$\Omega'_\delta = \Omega \setminus \overline{\Omega_\delta} = \{x \in \Omega : d(x) > \delta\},$$

i.e., $\operatorname{supp}\varphi \subset \Omega'_\delta$.
Hence, the difference quotient

$$\delta_h\varphi(x) \stackrel{\text{def}}{=} \frac{\varphi(x+h)-\varphi(x)}{|h|}, \quad x \in \Omega, \tag{31}$$

satisfies $\delta_h\varphi \in C^1_0(\Omega)$. Multiply eq. (26) by $\delta_h\varphi$ and integrate over Ω,

$$\int_\Omega \mathbf{a}(\nabla u)\cdot\nabla(\delta_h\varphi)\,\mathrm{d}x = \int_\Omega f\,(\delta_h\varphi)\,\mathrm{d}x$$

with the notation $\mathbf{a}(\nabla u) = |\nabla u|^{p-2}\nabla u$.
A simple substitution above yields

$$\begin{aligned}&\int_\Omega \frac{\mathbf{a}(\nabla u(x+h))-\mathbf{a}(\nabla u(x))}{|h|}\cdot\nabla\varphi(x)\,\mathrm{d}x\\ &= \int_\Omega \frac{f(x+h)-f(x)}{|h|}\,\varphi\,\mathrm{d}x\,.\end{aligned} \tag{32}$$

Now we use the Taylor formula

$$\begin{aligned}&\mathbf{a}(\nabla u(x+h))-\mathbf{a}(\nabla u(x))\\ &= \tilde{\mathbf{A}}(x;h)\,(\nabla u(x+h)-\nabla u(x))\end{aligned} \tag{33}$$

with the abbreviation

$$\tilde{\mathbf{A}}(x;h) \stackrel{\mathsf{def}}{=} \int_0^1 \mathbf{A}((1-s)\nabla u(x+h)+s\nabla u(x))\,\mathrm{d}s \in \mathbb{R}^{N\times N}$$

and replace the function φ by $(\delta_h u)\,\varphi$ in (32),

$$\begin{aligned} &\int_\Omega \left\langle \tilde{\mathbf{A}}(x;h)\,\delta_h(\nabla u),\, \delta_h(\nabla u)\right\rangle \varphi\,\mathrm{d}x \\ (34)\qquad &+ \int_\Omega \left\langle \tilde{\mathbf{A}}(x;h)\,\delta_h(\nabla u),\, \nabla\varphi\right\rangle (\delta_h u)\,\mathrm{d}x \\ &= \int_\Omega (\delta_h f)\,(\delta_h u)\,\varphi\,\mathrm{d}x\,. \end{aligned}$$

We used $\nabla[(\delta_h u)\varphi] = \varphi\cdot\delta_h(\nabla u) + (\delta_h u)\cdot\nabla\varphi$. We estimate the integrals on the left-hand side in (34) by the inequalities (30), combined with Cauchy's inequality, and abbreviate

$$(35)\quad \tilde{a}(x;h) \stackrel{\mathsf{def}}{=} \int_0^1 |(1-s)\nabla u(x+h)+s\nabla u(x)|^{p-2}\,\mathrm{d}s\,,$$

in order to get, with $\varphi = \varphi_\delta$ from (28),

$$\begin{aligned} &\gamma\int_\Omega \tilde{a}(x;h)\,|\delta_h(\nabla u)|^2\varphi_\delta\,\mathrm{d}x \\ (36)\qquad &\le \Gamma\int_{\mathcal{O}_\delta} \tilde{a}(x;h)\,|\delta_h(\nabla u)|\,|\nabla\varphi_\delta|\,|\delta_h u|\,\mathrm{d}x \\ &+ \int_\Omega |\delta_h f|\,|\delta_h u|\,\varphi_\delta\,\mathrm{d}x\,. \end{aligned}$$

To estimate the first integral on the right-hand side in ineq. (36), we use

$$(37)\qquad |\nabla\varphi_\delta(x)|^2 \le C\delta^{-2}\,\varphi_\delta(x)\,,\quad x\in\Omega\setminus\partial\mathcal{O}_\delta\,,$$

then apply Cauchy's inequality, thus arriving at

$$
\begin{aligned}
&\gamma \int_\Omega \tilde{a}(x;h)\,|\delta_h(\nabla u)|^2 \varphi_\delta \,\mathrm{d}x \qquad\qquad \text{(by (37))}\\
&\le \Gamma\, C^{1/2}\,\delta^{-1} \int_{\mathcal{O}_\delta} \tilde{a}(x;h)\,|\delta_h(\nabla u)| \cdot \varphi_\delta^{1/2} \cdot |\delta_h u| \,\mathrm{d}x\\
&+ \int_\Omega |\delta_h f|\,|\delta_h u|\,\varphi_\delta \,\mathrm{d}x\\
&\le \Gamma\, C^{1/2}\,\delta^{-1} \left(\int_{\mathcal{O}_\delta} \tilde{a}(x;h)\,|\delta_h(\nabla u)|^2 \varphi_\delta \,\mathrm{d}x\right)^{1/2}\\
&\times \left(\int_{\mathcal{O}_\delta} \tilde{a}(x;h)\,|\delta_h u|^2 \,\mathrm{d}x\right)^{1/2}\\
&+ \int_\Omega |\delta_h f|\,|\delta_h u|\,\varphi_\delta \,\mathrm{d}x\\
&\le \frac{\gamma}{2} \int_{\mathcal{O}_\delta} \tilde{a}(x;h)\,|\delta_h(\nabla u)|^2 \varphi_\delta \,\mathrm{d}x\\
&+ \frac{1}{2}\gamma^{-1}\,\Gamma^2\, C\,\delta^{-2} \int_{\mathcal{O}_\delta} \tilde{a}(x;h)\,|\delta_h u|^2 \,\mathrm{d}x\\
&+ \int_\Omega |\delta_h f|\,|\delta_h u|\,\varphi_\delta \,\mathrm{d}x
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\int_\Omega \tilde{a}(x;h)\,|\delta_h(\nabla u)|^2 \varphi_\delta \,\mathrm{d}x \le\\
&C\,(\Gamma/(\gamma\delta))^2 \int_{\mathcal{O}_\delta} \tilde{a}(x;h)\,|\delta_h u|^2 \,\mathrm{d}x\\
&+ (2/\gamma) \int_\Omega |\delta_h f|\,|\delta_h u|\,\varphi_\delta \,\mathrm{d}x\,.
\end{aligned}
$$

We state a few geometric inequalities

(Takáč, IUMJ 2002, Lemma A.1, p. 233).

Let $1 < p < \infty$ and $p \neq 2$. Assume that $\Theta \in L^\infty(0,1)$ satisfies $\Theta \geq 0$ in $(0,1)$ and $T = \int_0^1 \Theta(s)\,\mathrm{d}s > 0$. Then there is a constant $c_p \equiv c_p(\Theta) > 0$ such that, for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$:

If $p > 2$ then

(38)

$$c_p(\Theta)^{p-2} \left(\max_{0 \leq s \leq 1} |\mathbf{a} + s\mathbf{b}| \right)^{p-2} \leq \int_0^1 |\mathbf{a} + s\mathbf{b}|^{p-2}\, \Theta(s)\,\mathrm{d}s$$

$$\leq T \cdot \left(\max_{0 \leq s \leq 1} |\mathbf{a} + s\mathbf{b}| \right)^{p-2},$$

and if $1 < p < 2$ and $|\mathbf{a}| + |\mathbf{b}| > 0$ then

(39)

$$T \cdot \left(\max_{0 \leq s \leq 1} |\mathbf{a} + s\mathbf{b}| \right)^{p-2} \leq \int_0^1 |\mathbf{a} + s\mathbf{b}|^{p-2}\, \Theta(s)\,\mathrm{d}s$$

$$\leq c_p(\Theta)^{p-2} \left(\max_{0 \leq s \leq 1} |\mathbf{a} + s\mathbf{b}| \right)^{p-2}.$$

Equivalently, in both cases ($p \neq 2$), the ratio

$$\int_0^1 |\mathbf{a} + s\mathbf{b}|^{p-2}\, \Theta(s)\,\mathrm{d}x \Bigg/ \left(\max_{0 \leq s \leq 1} |\mathbf{a} + s\mathbf{b}| \right)^{p-2}$$

is bounded below and above by positive constants, whenever $|\mathbf{a}| + |\mathbf{b}| > 0$.

Now we apply inequalities (38) and (39) to the expression $\tilde{a}(x;h)$ (defined in (35)) in order to conclude that there exist constants $C_1', C_2' > 0$ such that

(40)
$$\int_\Omega \hat{a}(x;h)\, |\delta_h(\nabla u)|^2 \varphi_\delta \, \mathrm{d}x \leq$$
$$C_1' \int_{\mathcal{O}_\delta} \hat{a}(x;h)\, |\delta_h u|^2 \, \mathrm{d}x + C_2' \int_\Omega |\delta_h f|\, |\delta_h u|\, \varphi_\delta \, \mathrm{d}x \,,$$

where we have abbreviated

(41)
$$\hat{a}(x;h) \stackrel{\mathrm{def}}{=} \left(\max_{0\leq s\leq 1} |(1-s)\nabla u(x+h) + s\nabla u(x)| \right)^{p-2} .$$

Now we recall our hypothesis $\int_\Omega |\nabla f|\, \mathrm{d}x < \infty$ and the regularity result $|\nabla u| \leq \mathrm{const} < \infty$ in Ω, together with hypothesis (27) if $1 < p < 2$. Applying these inequalities to the right-hand side of (40), we arrive at

$$\int_\Omega \hat{a}(x;h)\, |\delta_h(\nabla u)|^2 \varphi_\delta \, \mathrm{d}x \leq C' < \infty \qquad (42)$$

where the constant $C' > 0$ is independent from $h \in \mathbb{R}^N$ with $0 < |h| < \delta$.

Finally, we deduce from (29) that the Jacobian matrix $\mathbf{B} = (B_{ij})_{i,j=1}^N$ of the mapping $\boldsymbol{\eta} \mapsto \mathbf{b}(\boldsymbol{\eta}) =$

$|\boldsymbol{\eta}|^{(p/2)-1}\boldsymbol{\eta} : \mathbb{R}^N \to \mathbb{R}^N$ equals

(43)

$$\mathbf{B}(\boldsymbol{\eta}) = |\boldsymbol{\eta}|^{(p/2)-1}\left(\mathbf{I} + \frac{p-2}{2}\,\frac{\boldsymbol{\eta}\otimes\boldsymbol{\eta}}{|\boldsymbol{\eta}|^2}\right), \quad \boldsymbol{\eta}\in\mathbb{R}^N\setminus\{\mathbf{0}\}\,.$$

In analogy with the Taylor formula in (33),

$$\begin{aligned} &\mathbf{b}(\nabla u(x+h)) - \mathbf{b}(\nabla u(x)) \\ &= \tilde{\mathbf{B}}(x;h)\,(\nabla u(x+h) - \nabla u(x)) \end{aligned} \tag{44}$$

with the abbreviation

$$\tilde{\mathbf{B}}(x;h) \stackrel{\text{def}}{=} \int_0^1 \mathbf{B}((1-s)\nabla u(x+h) + s\nabla u(x))\,\mathrm{d}s \in \mathbb{R}^{N\times N}\,.$$

Now we treat the L^2-norm

$$\begin{aligned} &\int_\Omega |\delta_h(\mathbf{b}(\nabla u))|^2\,\varphi_\delta\,\mathrm{d}x \\ &= \int_\Omega \left\langle \tilde{\mathbf{B}}(x;h)\,\delta_h(\nabla u),\, \tilde{\mathbf{B}}(x;h)\,\delta_h(\nabla u)\right\rangle \varphi_\delta\,\mathrm{d}x \\ &= \int_\Omega \left\langle \tilde{\mathbf{B}}(x;h)^2\,\delta_h(\nabla u),\, \delta_h(\nabla u)\right\rangle \varphi_\delta\,\mathrm{d}x \end{aligned} \tag{45}$$

where, using the abbreviation

$$\boldsymbol{\eta}(s) = (1-s)\nabla u(x+h) + s\nabla u(x) \in \mathbb{R}^N\,, \quad 0\le s\le 1\,,$$

we have

(46)

$$\tilde{\mathbf{B}}(x;h)^2 = \int_0^1\int_0^1 \mathbf{B}(\boldsymbol{\eta}(s))\,\mathbf{B}(\boldsymbol{\eta}(t))\,\mathrm{d}s\,\mathrm{d}t =$$
$$\int_0^1\int_0^1 |\boldsymbol{\eta}(s)|^{(p/2)-1}\,|\boldsymbol{\eta}(t)|^{(p/2)-1}\,\mathbf{C}(\boldsymbol{\eta}(s),\boldsymbol{\eta}(t))\,\mathrm{d}s\,\mathrm{d}t$$

with the $(N \times N)$-matrix

$$\mathbf{C}(\mathbf{a},\mathbf{b}) = \left(\mathbf{I} + \frac{p-2}{2}\,\frac{\mathbf{a}\otimes\mathbf{a}}{|\mathbf{a}|^2}\right)\left(\mathbf{I} + \frac{p-2}{2}\,\frac{\mathbf{b}\otimes\mathbf{b}}{|\mathbf{b}|^2}\right) \in \mathbb{R}^{N\times N}$$

being uniformly bounded for $\mathbf{a},\mathbf{b} \in \mathbb{R}^N \setminus \{0\}$, cf. (43). Furthermore, $\mathbf{C}(\mathbf{a},\mathbf{b}) = \mathbf{C}(\mathbf{a})\,\mathbf{C}(\mathbf{b})$ where

$$\mathbf{C}(\mathbf{a}) \stackrel{\text{def}}{=} \left(\mathbf{I} + \frac{p-2}{2}\,\frac{\mathbf{a}\otimes\mathbf{a}}{|\mathbf{a}|^2}\right) \in \mathbb{R}^{N\times N}$$

is a symmetric matrix with the eigenvalues 1 and $p/2$ (if $N \geq 2$). Consequently, there is a constant $\Gamma' > 0$ such that the kernel
$\mathbf{B}(\boldsymbol{\eta}(s))\,\mathbf{B}(\boldsymbol{\eta}(t))$ of the quadratic form contained in the integrand on the right-hand side of eq. (45) satisfies

$$\begin{aligned}
&\langle \mathbf{B}(\boldsymbol{\eta}(s))\,\mathbf{B}(\boldsymbol{\eta}(t))\,\boldsymbol{\xi},\,\boldsymbol{\xi}\rangle = \langle \mathbf{B}(\boldsymbol{\eta}(t))\,\boldsymbol{\xi},\,\mathbf{B}(\boldsymbol{\eta}(s))\,\boldsymbol{\xi}\rangle \\
&\leq |\mathbf{B}(\boldsymbol{\eta}(t))\,\boldsymbol{\xi}| \cdot |\mathbf{B}(\boldsymbol{\eta}(s))\,\boldsymbol{\xi}| \\
&\leq |\mathbf{B}(\boldsymbol{\eta}(t))| \cdot |\mathbf{B}(\boldsymbol{\eta}(s))| \cdot |\boldsymbol{\xi}|^2 \\
&\leq \Gamma'\,|\boldsymbol{\eta}(t)|^{(p/2)-1}\,|\boldsymbol{\eta}(s)|^{(p/2)-1}\,|\boldsymbol{\xi}|^2
\end{aligned}$$

for $\boldsymbol{\xi} \in \mathbb{R}^N$ and $s,t \in [0,1]$ with $\boldsymbol{\eta}(s),\boldsymbol{\eta}(t) \neq 0$. We first integrate this inequality with respect to $s,t \in [0,1]$, then apply it to (46) to get
(47)

$$\left\langle \tilde{\mathbf{B}}(x;h)^2\,\boldsymbol{\xi},\,\boldsymbol{\xi}\right\rangle \leq \Gamma'\,\tilde{b}(x;h)^2\,|\boldsymbol{\xi}|^2 \qquad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^N,$$

with the abbreviation

$$\tilde{b}(x;h) \stackrel{\text{def}}{=} \int_0^1 |\boldsymbol{\eta}(s)|^{(p/2)-1}\,\mathrm{d}s = \int_0^1 |(1-s)\nabla u(x+h) + s\nabla u(x)|^{(p/2)-1}\,\mathrm{d}s\,,$$

in analogy with (35). This integral is estimated from above by inequalities (38) and (39),

$$\tilde{b}(x;h) \leq C_p\,\hat{b}(x;h)\,, \tag{48}$$

where $C_p > 0$ is a numerical constant depending only on p ($1 < p < \infty$) and

$$\hat{b}(x;h) \stackrel{\text{def}}{=} \left(\max_{0\leq s\leq 1} |(1-s)\nabla u(x+h) + s\nabla u(x)| \right)^{(p/2)-1}.$$

We combine inequalities (47) and (48) and apply them to the quadratic form contained in the integrand on the right-hand side of eq. (45),

$$\int_\Omega |\delta_h(\mathbf{b}(\nabla u))|^2\,\varphi_\delta\,\mathrm{d}x \leq \Gamma' \int_\Omega \tilde{b}(x;h)^2\,|\delta_h(\nabla u)|^2 \varphi_\delta\,\mathrm{d}x\,, \leq \Gamma'' \int_\Omega \hat{b}(x;h)^2\,|\delta_h(\nabla u)|^2 \varphi_\delta\,\mathrm{d}x\,, \tag{49}$$

with the constant $\Gamma'' = \Gamma'\,C_p^2 > 0$. As $\hat{a}(x;h) = \hat{b}(x;h)^2$ by (41), ineq. (42) implies

$$\int_\Omega |\delta_h(\mathbf{b}(\nabla u))|^2\,\varphi_\delta\,\mathrm{d}x \leq C'' < \infty \tag{50}$$

where the constant $C'' > 0$ is independent from $h \in \mathbb{R}^N$ with $0 < |h| < \delta$. We are now ready to derive all conclusions of our proposition from this estimate.
■

Proof of **Proposition 1.** (Pohozhaev's id.)

$$\frac{N-p}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega f(x)\,(x\cdot\nabla u)\,\mathrm{d}x$$
$$= \frac{1}{p}\int_{\partial\Omega} |\nabla u|^p\,(x\cdot\nu(x))\,\mathrm{d}\sigma(x)$$
$$- \int_{\partial\Omega} |\nabla u|^{p-2}\,(x\cdot\nabla u)(\nu(x)\cdot\nabla u)\,\mathrm{d}\sigma(x)\,.$$

We begin with the formal calculations

$$\begin{aligned} &\operatorname{div}\Big((x\cdot\nabla u)\,|\nabla u|^{p-2}\nabla u\Big)\\ &= (x\cdot\nabla u)\,\Delta_p u\\ &+ \nabla(x\cdot\nabla u)\cdot|\nabla u|^{p-2}\nabla u\\ &= (x\cdot\nabla u)\,\Delta_p u + |\nabla u|^p + \frac{1}{p}\,x\cdot\nabla|\nabla u|^p\,, \end{aligned} \tag{51}$$

$$\operatorname{div}(x\,|\nabla u|^p) = N\,|\nabla u|^p + x\cdot\nabla|\nabla u|^p \tag{52}$$

which are valid pointwise in Ω if $u\in C^2(\Omega)$.

From eq. (51) we subtract $(1/p)$-multiple of eq. (52) and observe that the difference

(53)

$$\operatorname{div}\mathbf{v}(x) = \operatorname{div}\Big((x\cdot\nabla u)\,|\nabla u|^{p-2}\nabla u\Big) - \frac{1}{p}\operatorname{div}(x\,|\nabla u|^p)$$
$$= (x\cdot\nabla u)\,\Delta_p u + \left(1-\frac{N}{p}\right)|\nabla u|^p$$
$$= -f(x)\,(x\cdot\nabla u) - \frac{N-p}{p}\,|\nabla u|^p$$

belongs to $L^1(\Omega)$, with the vector field

$$\mathbf{v}(x) \stackrel{\text{def}}{=} (x \cdot \nabla u)\, |\nabla u|^{p-2} \nabla u - \frac{1}{p}\, x\, |\nabla u|^p$$

being continuous in $\overline{\Omega}$, i.e., $\mathbf{v} \in [C(\overline{\Omega})]^N$. We complete the proof of Pohozhaev's identity by applying the divergence theorem to eq. (53). ■