Rheometrical flow systems

Part 2. Theory for the orthogonal rheometer, including an exact solution of the Navier–Stokes equations

By T. N. G. ABBOTT and K. WALTERS

Department of Applied Mathematics, University College of Wales, Aberystwyth

Consideration is given to the flow of viscous and elastico-viscous liquids contained between two infinite parallel plates which rotate with the same angular velocity Ω about axes normal to the plates but not coincident. In the viscous case, it is shown that a simple exact solution of the relevant equations can be obtained. In the elastico-viscous case, certain formulae are derived which should facilitate the interpretation of experimental results obtained from the Maxwell orthogonal rheometer.

1. Introduction

The orthogonal rheometer is a new instrument which was introduced by Maxwell & Chartoff (1965) to determine the complex dynamic viscosity η^* of an elastico-viscous liquid. The instrument consists essentially of two flat parallel plates which rotate with the same angular velocity Ω about two axes normal to the plates but not coincident. The distance between the axes is small. The test fluid is contained between the plates, and the two components of the tangential force on one of the plates are measured. From these forces, Maxwell & Chartoff claim that η^* can be determined. An early theoretical attempt to justify this claim (Blyler & Kurtz 1967) was concerned with a very simple equation of state, and the analysis indicated that the complex viscosity could be determined without any restriction on the distance between the rotating axes. A later and more thorough analysis by Bird & Harris (1968) indicated that a limiting procedure in terms of the distance between the rotating axes is required to determine η^*. The analyses of Blyler & Kurtz (1967) and Bird & Harris (1968) were both limited to situations where the effects of fluid inertia could be ignored.

In the present paper, we consider the flow generated in the Maxwell orthogonal rheometer for situations where fluid inertia is not negligible. In keeping with the earlier analyses, we base the theory on the assumption that the fluid is contained between *infinite* plates, although in practice the test fluid occupies a finite volume, i.e. we shall ignore edge effects.

2. Solution for a Newtonian liquid

We first discuss the problem for a Newtonian viscous liquid. This initial study, apart from indicating the method of solution for the elastico-viscous case, also leads to an exact solution of the Navier–Stokes equations. Since such solutions are rare, we thought it of interest to discuss this case separately, especially in view of the fact that it is not necessary to make any restriction on the distance between the axes of rotation in this case.

All physical quantities will be referred to cylindrical polar co-ordinates (r, θ, z), the z axis being placed symmetrically between the two axes of rotation (figure 1).
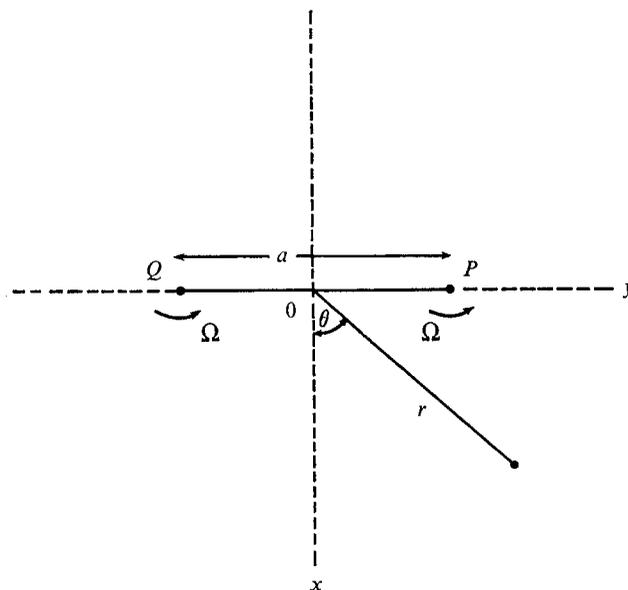

FIGURE 1

The upper plate, which is given by $z = h$, rotates about an axis through P, and the bottom plate given by $z = 0$ rotates about an axis through Q. The distance between P and Q is a. In §3, we shall find it convenient to restrict the discussion to small values of a. This is not necessary in the viscous case.

If the physical components of the velocity vector are given by $v_{(r)}$, $v_{(\theta)}$, $v_{(z)}$, the appropriate boundary conditions are

$$\left.\begin{aligned} v_{(r)} &= -\tfrac{1}{2}\Omega a \cos\theta, & v_{(\theta)} &= \Omega[r + \tfrac{1}{2}a \sin\theta], & v_{(z)} &= 0 \quad \text{on} \quad z = 0, \\ v_{(r)} &= \tfrac{1}{2}\Omega a \cos\theta, & v_{(\theta)} &= \Omega[r - \tfrac{1}{2}a \sin\theta], & v_{(z)} &= 0 \quad \text{on} \quad z = h. \end{aligned}\right\} \tag{1}$$

We note that these boundary conditions are exact and are not restricted to small values of a.

The relevant equations for a Newtonian liquid are the Navier–Stokes equations given by (2)–(4) and the equation of continuity (5).

$$\rho\left[\frac{\partial v_{(r)}}{\partial t}+v_{(r)}\frac{\partial v_{(r)}}{\partial r}+\frac{v_{(\theta)}}{r}\frac{\partial v_{(r)}}{\partial \theta}+v_{(z)}\frac{\partial v_{(r)}}{\partial z}-\frac{v_{(\theta)}^2}{r}\right]$$
$$=-\frac{\partial p}{\partial r}+\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_{(r)}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_{(r)}}{\partial \theta^2}+\frac{\partial^2 v_{(r)}}{\partial z^2}-\frac{v_{(r)}}{r^2}-\frac{2}{r^2}\frac{\partial v_{(\theta)}}{\partial \theta}\right], \qquad (2)$$

$$\rho\left[\frac{\partial v_{(\theta)}}{\partial t}+v_{(r)}\frac{\partial v_{(\theta)}}{\partial r}+\frac{v_{(\theta)}}{r}\frac{\partial v_{(\theta)}}{\partial \theta}+v_{(z)}\frac{\partial v_{(\theta)}}{\partial z}+\frac{v_{(r)}v_{(\theta)}}{r}\right]$$
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_{(\theta)}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_{(\theta)}}{\partial \theta^2}+\frac{\partial^2 v_{(\theta)}}{\partial z^2}+\frac{2}{r^2}\frac{\partial v_{(r)}}{\partial \theta}-\frac{v_{(\theta)}}{r^2}\right], \qquad (3)$$

$$\rho\left[\frac{\partial v_{(z)}}{\partial t}+v_{(r)}\frac{\partial v_{(z)}}{\partial r}+\frac{v_{(\theta)}}{r}\frac{\partial v_{(z)}}{\partial \theta}+v_{(z)}\frac{\partial v_{(z)}}{\partial z}\right]$$
$$=-\rho g-\frac{\partial p}{\partial z}+\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_{(z)}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_{(z)}}{\partial \theta^2}+\frac{\partial^2 v_{(z)}}{\partial z^2}\right], \qquad (4)$$

$$\frac{\partial}{\partial r}(rv_{(r)})+\frac{\partial}{\partial \theta}(v_{(\theta)})+\frac{\partial}{\partial z}(rv_{(z)})=0, \qquad (5)$$

where p is an isotropic pressure, ρ the density of the fluid, g the acceleration due to gravity and η is the (constant) dynamic viscosity.

The boundary conditions (1) suggest a velocity distribution of the form,

$$\left.\begin{aligned}
v_{(r)} &= A(z)\cos\theta+B(z)\sin\theta, \\
v_{(\theta)} &= \Omega r+B(z)\cos\theta-A(z)\sin\theta, \\
v_{(z)} &= 0,
\end{aligned}\right\} \qquad (6)$$

which automatically satisfies the equation of continuity. In any plane $z = $ constant, (6) is equivalent to adding a translational velocity, with components $(A(z), B(z))$ with respect to the (x, y) axes, to the rotation about the z axis.

The boundary conditions (1) require

$$\left.\begin{aligned}
A(0) &= -\tfrac{1}{2}\Omega a, & B(0) &= 0, \\
A(h) &= \tfrac{1}{2}\Omega a, & B(h) &= 0.
\end{aligned}\right\} \qquad (7)$$

Substituting (6) into the equations of motion (2)–(4), we obtain

$$-\rho\Omega^2 r+\rho\Omega[A\sin\theta-B\cos\theta]=-(\partial p/\partial r)+\eta[A''\cos\theta+B''\sin\theta], \qquad (8)$$

$$\rho\Omega[A\cos\theta+B\sin\theta]=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\eta[B''\cos\theta-A''\sin\theta], \qquad (9)$$

$$\rho g=-\frac{\partial p}{\partial z}, \qquad (10)$$

where the dashes refer to differentiation with respect to z. We note with interest that (8)–(10) do not contain any non-linear terms in $A(z)$ and $B(z)$.

Eliminating p between (8)–(10), we obtain

$$\rho\Omega[A'\sin\theta-B'\cos\theta]=\eta[A'''\cos\theta+B'''\sin\theta], \qquad (11)$$

and $$\rho\Omega[A'\cos\theta + B'\sin\theta] = \eta[B'''\cos\theta - A'''\sin\theta], \tag{12}$$

which imply $$A''' + (\rho\Omega/\eta)B' = 0, \tag{13}$$

$$B''' - (\rho\Omega/\eta)A' = 0. \tag{14}$$

Equations (13) and (14) can be integrated immediately to give

$$A'' + (\rho\Omega/\eta)B = q, \tag{15}$$

$$B'' - (\rho\Omega/\eta)A = s, \tag{16}$$

where q and s are arbitrary constants. From (15) and (16), we obtain

$$A^{\mathrm{iv}} + (\rho^2\Omega^2/\eta^2)A = (-\Omega\rho/\eta)s, \tag{17}$$

$$B^{\mathrm{iv}} + (\rho^2\Omega^2/\eta^2)B = (\Omega\rho/\eta)q. \tag{18}$$

Equations (17) and (18) have to be solved subject to

$$\left.\begin{array}{llll} A(0) = -\tfrac{1}{2}\Omega a, & A''(0) = q, & A(h) = \tfrac{1}{2}\Omega a, & A''(h)^{\cdot} = q, \\ B(0) = 0, & B''(0) = (-\rho\Omega^2 a/2\eta) + s, & B(h) = 0, & B''(h) = (\rho\Omega^2 a/2\eta) + s. \end{array}\right\} \tag{19}$$

The corresponding pressure distribution is

$$p = p_0 + (\rho\Omega^2\tfrac{1}{2}r^2) - \rho g z + \eta[qr\cos\theta + sr\sin\theta], \tag{20}$$

where p_0 is a constant. We see from (20) that non-zero values of q and s would give rise to a pressure gradient between the plates with a corresponding 'Poiseuille-type' flow. In order to remove the possibility of this component (and at the same time ensure that the velocity distribution is symmetrical in the $z = \tfrac{1}{2}h$ plane) we take $q = s = 0$. In this case the solution is given by (6) with

$$A(z) = (\tfrac{1}{2}\Omega a)[\{\sinh kz\cos kz + \sinh k(z-h)\cos k(z-h)\}\sinh kh\cos kh$$
$$+ \{\cosh kz\sin kz + \cosh k(z-h)\sin k(z-h)\}\cosh kh\sin kh]/[\sinh^2 kh\cos^2 kh$$
$$+ \cosh^2 kh\sin^2 kh], \tag{21}$$

$$B(z) = (\tfrac{1}{2}\Omega a)[\{\cosh kz\sin kz + \cosh k(z-h)\sin k(z-h)\}\sinh kh\cos kh$$
$$- \{\sinh kz\cos kz + \sinh k(z-h)\cos k(z-h)\}\cosh kh\sin kh]/[\sinh^2 kh\cos^2 kh$$
$$+ \cosh^2 kh\sin^2 kh], \tag{22}$$

where $$k = (\rho\Omega/2\eta)^{\frac{1}{2}}. \tag{23}$$

The solution implies that each plane $z = $ constant moves as if rigid with angular velocity Ω about a point, but the locus of these points as z varies is not a straight line joining the centres of the two planes. Figure 2 contains the projection of this locus on the x, y plane for various values of kh.

3. Solution for an elastico-viscous liquid

In the case of an *elastico*-viscous liquid, it is necessary to assume that the distance a between the axes of rotation is small, so that a solution can be obtained by expanding the relevant physical variables as power series in a. The basic reason for this approximation is mathematical, in that the complicated (non-linear) equations of state for elastico-viscous liquids introduce non-linearities into the relevant equations, which are not present in the viscous case. However,

Bird & Harris (1968) have shown that such an approximation is also required if the complex dynamic viscosity is to be obtained from experimental data and these authors conclude that the experimental results of Maxwell & Chartoff (1965) were taken at sufficiently small a for effects of order a^2 to be negligible. Restricting the discussion to very small values of a is therefore reasonable as well as necessary.
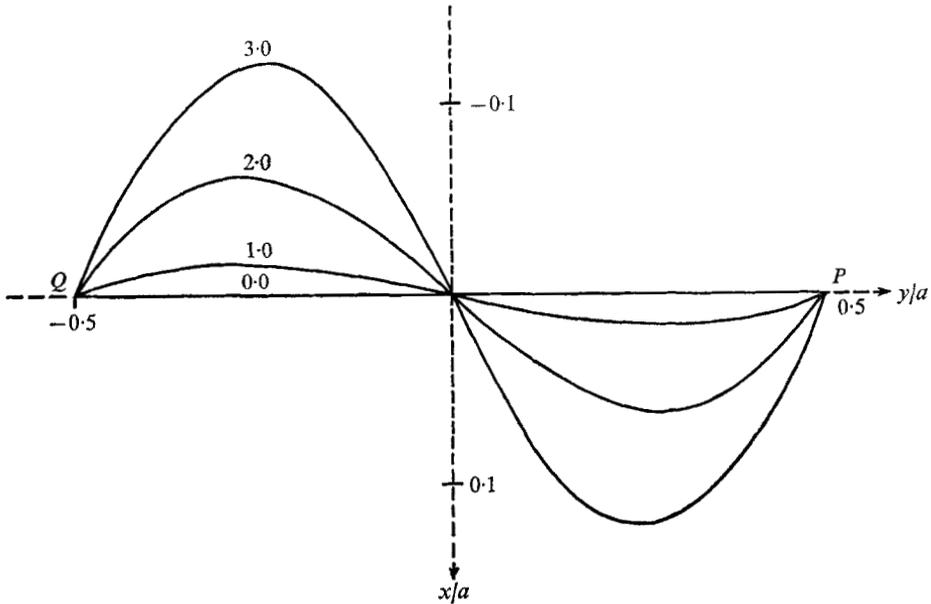

FIGURE 2. The projection of the locus of the centres of rotation on the x, y plane for various values of kh.

The basic equations to be considered are the equation of continuity (5), the stress equations of motion given by (24)–(26) below and the equations of state.

$$\rho\left[\frac{\partial v_{(r)}}{\partial t} + v_{(r)}\frac{\partial v_{(r)}}{\partial r} + \frac{v_{(\theta)}}{r}\frac{\partial v_{(r)}}{\partial \theta} + v_{(z)}\frac{\partial v_{(r)}}{\partial z} - \frac{v_{(\theta)}^2}{r}\right]$$
$$= \frac{\partial p_{(rr)}}{\partial r} + \frac{1}{r}\frac{\partial p_{(r\theta)}}{\partial \theta} + \frac{\partial p_{(rz)}}{\partial z} - \frac{p_{(\theta\theta)}}{r} + \frac{p_{(rr)}}{r}, \quad (24)\dagger$$

$$\rho\left[\frac{\partial v_{(\theta)}}{\partial t} + v_{(r)}\frac{\partial v_{(\theta)}}{\partial r} + \frac{v_{(\theta)}}{r}\frac{\partial v_{(\theta)}}{\partial \theta} + v_{(z)}\frac{\partial v_{(\theta)}}{\partial z} + \frac{v_{(r)}v_{(\theta)}}{r}\right]$$
$$= \frac{\partial p_{(r\theta)}}{\partial r} + \frac{1}{r}\frac{\partial p_{(\theta\theta)}}{\partial \theta} + \frac{\partial p_{(\theta z)}}{\partial z} + \frac{2p_{(r\theta)}}{r}, \quad (25)$$

$$\rho\left[\frac{\partial v_{(z)}}{\partial t} + v_{(r)}\frac{\partial v_{(z)}}{\partial r} + \frac{v_{(\theta)}}{r}\frac{\partial v_{(z)}}{\partial \theta} + v_{(z)}\frac{\partial v_{(z)}}{\partial z}\right] + \rho g$$
$$= \frac{\partial p_{(zr)}}{\partial r} + \frac{1}{r}\frac{\partial p_{(z\theta)}}{\partial \theta} + \frac{\partial p_{(zz)}}{\partial z} + \frac{p_{(zr)}}{r}, \quad (26)$$

where p_{ik} is the stress tensor.

† Parentheses placed round suffixes denote physical components of tensors.

When the distance between the axes of rotation is small, the deformation experienced by individual fluid elements is also small, and if terms of order a^2 are neglected, we may take the equations of state in the form (cf. Walters 1970)

$$p_{ik} = -pg_{ik} + p'_{ik}, \tag{27}$$

$$p'_{ik} = \int_{-\infty}^{t} M_1(t-t') C_{ik}(t') \, dt', \tag{28}$$

where g_{ik} is the metric tensor of a suitable co-ordinate system x^i and C_{ik} is given by

$$C_{ik} \equiv \frac{\partial x'^m}{\partial x^i} \frac{\partial x'^l}{\partial x^k} g_{ml}(\mathbf{x'}) - g_{ik}(\mathbf{x}), \tag{29}$$

x'^i being the position at time t' of the element that is instantaneously at the point x^i at time t. We note that the relation between the kernel function M_1 and the complex dynamic viscosity η^* is

$$\eta^* = \eta' - (iG'/\Omega) = -(1/i\Omega) \int_0^\infty M_1(\xi)[1 - e^{-i\Omega\xi}] \, d\xi, \tag{30}$$

η' being known as the dynamic viscosity and G' the dynamic rigidity.

Equations (5), (24)–(28) have to be solved subject to the boundary conditions (1). In the light of these boundary conditions, we are led to consider a velocity distribution of the form,

$$\left. \begin{aligned} v_{(r)} &= au(r,z)e^{i\theta}, \\ v_{(\theta)} &= r\Omega + av(r,z)e^{i\theta}, \\ v_{(z)} &= aw(r,z)e^{i\theta}, \end{aligned} \right\} \tag{31}$$

where u, v and w may be complex and the real part is implied. In the following, we shall work to first order in a.

The displacement functions x'^i, which we shall write as r', θ', z', are given by (Oldroyd 1950)

$$\left. \begin{aligned} \frac{\partial r'}{\partial t} + v_{(r)}\frac{\partial r'}{\partial r} + \frac{v_{(\theta)}}{r}\frac{\partial r'}{\partial \theta} + v_{(z)}\frac{\partial r'}{\partial z} &= 0, \\ \frac{\partial \theta'}{\partial t} + v_{(r)}\frac{\partial \theta'}{\partial r} + \frac{v_{(\theta)}}{r}\frac{\partial \theta'}{\partial \theta} + v_{(z)}\frac{\partial \theta'}{\partial z} &= 0, \\ \frac{\partial z'}{\partial t} + v_{(r)}\frac{\partial z'}{\partial r} + \frac{v_{(\theta)}}{r}\frac{\partial z'}{\partial \theta} + v_{(z)}\frac{\partial z'}{\partial z} &= 0. \end{aligned} \right\} \tag{32}$$

The solution of (32) subject to $r' = r$, $\theta' = \theta$, $z' = z$ when $t' = t$, is

$$\left. \begin{aligned} r' &= r + \frac{iau}{\Omega} e^{i\theta}[1 - e^{-i\Omega(t-t')}], \\ \theta' &= \theta - \Omega(t-t') + \frac{iav}{\Omega r} e^{i\theta}[1 - e^{-i\Omega(t-t')}], \\ z' &= z + \frac{iaw}{\Omega} e^{i\theta}[1 - e^{-i\Omega(t-t')}]. \end{aligned} \right\} \tag{33}$$

From (29) and (33), it can be shown that

$$
\left.
\begin{aligned}
C_{rr}(r',\theta',z') &= \frac{2ia}{\Omega}\,[1-e^{-i\Omega(t-t')}]\,e^{i\theta}\,\frac{\partial u}{\partial r}, \\[4pt]
C_{zz}(r',\theta',z') &= \frac{2ia}{\Omega}\,[1-e^{-i\Omega(t-t')}]\,e^{i\theta}\,\frac{\partial w}{\partial z}, \\[4pt]
C_{\theta\theta}(r',\theta',z') &= \frac{2ia}{\Omega}\,[1-e^{-i\Omega(t-t')}]\,e^{i\theta}(irv+ru), \\[4pt]
C_{r\theta}(r',\theta',z') &= \frac{ia}{\Omega}\,[1-e^{-i\Omega(t-t')}]\,e^{i\theta}\left(iu+r^2\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right), \\[4pt]
C_{rz}(r',\theta',z') &= \frac{ia}{\Omega}\,[1-e^{-i\Omega(t-t')}]\,e^{i\theta}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right), \\[4pt]
C_{\theta z}(r',\theta',z') &= \frac{ia}{\Omega}\,[1-e^{-i\Omega(t-t')}]\,e^{i\theta}\left(r\frac{\partial v}{\partial z}+iw\right).
\end{aligned}
\right\}
\tag{34}
$$

We note from the form of the deformation tensor C_{ik} that individual fluid elements are subjected to a periodic deformation history although the flow is steady, i.e. $\partial/\partial t \equiv 0$ (cf. Walters 1970).

Substituting (34) into (27)–(28) and using (30), we obtain for the physical components of the stress tensor

$$
\left.
\begin{aligned}
p_{(rr)} &= -p+2\eta^* a e^{i\theta}\frac{\partial u}{\partial r}, \\[4pt]
p_{(\theta\theta)} &= -p+2\eta^* a e^{i\theta}\left[\frac{iv+u}{r}\right], \\[4pt]
p_{(zz)} &= -p+2\eta^* a e^{i\theta}\frac{\partial w}{\partial z}, \\[4pt]
p_{(r\theta)} &= \eta^* a e^{i\theta}\left[\frac{iu}{r}+r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right], \\[4pt]
p_{(rz)} &= \eta^* a e^{i\theta}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right], \\[4pt]
p_{(\theta z)} &= \eta^* a e^{i\theta}\left[\frac{\partial v}{\partial z}+\frac{iw}{r}\right].
\end{aligned}
\right\}
\tag{35}
$$

Substituting (31) and (35) into the stress equations of motion (24)–(26) and writing $p = p_0 - \rho g z + (\rho\Omega^2 r^2/2) + a\overline{p}e^{i\theta}$ we obtain

$$
\rho\Omega(iu-2v) = -\frac{\partial\overline{p}}{\partial r}+\eta^*\left[\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial z^2}-\frac{2iv}{r^2}-\frac{2u}{r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right],
\tag{36}
$$

$$
\rho\Omega(iv+2u) = \frac{-i\overline{p}}{r}+\eta^*\left[\frac{\partial^2 v}{\partial r^2}+\frac{\partial^2 v}{\partial z^2}+\frac{2iu}{r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{2v}{r^2}\right],
\tag{37}
$$

$$
\rho\Omega iw = -\frac{\partial\overline{p}}{\partial z}+\eta^*\left[\frac{\partial^2 w}{\partial z^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)-\frac{w}{r^2}\right].
\tag{38}
$$

We note that (36)–(38) are essentially the Navier–Stokes equations, with the complex dynamic viscosity η^* replacing the constant viscosity coefficient η (cf. (2)–(4)).

If we write (cf. §2)

$$u = u(z), \quad v = v(z), \quad w = 0, \quad \bar{p} = 0, \tag{39}$$

(38) is satisfied identically, and the equation of continuity implies

$$u + iv = 0. \tag{40}$$

Also, (36) and (37) yield
$$u'' - \alpha^2 u = 0, \tag{41}$$

$$v'' - \alpha^2 v = 0, \tag{42}$$

where $\alpha^2 = -i\Omega\rho/\eta^*$ (cf. Walters 1970). The appropriate boundary conditions are

$$\begin{aligned}
u &= -\tfrac{1}{2}\Omega, \quad v = -\tfrac{1}{2}i\Omega \quad \text{when} \quad z = 0, \\
u &= \tfrac{1}{2}\Omega, \qquad v = \tfrac{1}{2}i\Omega \qquad \text{when} \quad z = h.
\end{aligned} \tag{43}$$

The solutions of (41) and (42) subject to (43) are

$$\begin{aligned}
u &= (\tfrac{1}{2}\Omega/\sinh\alpha h)[\sinh\alpha z + \sinh\alpha(z-h)], \\
v &= (\tfrac{1}{2}i\Omega/\sinh\alpha h)[\sinh\alpha z + \sinh\alpha(z-h)].
\end{aligned} \tag{44}$$

The relevant stresses in the problem under consideration are the shear stresses $p_{(zr)}$ and $p_{(z\theta)}$, which are given by

$$p_{(zr)} = \text{Re}\left[-\frac{ia\rho\Omega^2\{\cosh\alpha z + \cosh\alpha(z-h)\}e^{i\theta}}{2\alpha\sinh\alpha h} \right], \tag{45}$$

$$p_{(z\theta)} = \text{Re}\left[\frac{a\rho\Omega^2\{\cosh\alpha z + \cosh\alpha(z-h)\}e^{i\theta}}{2\alpha\sinh\alpha h} \right]. \tag{46}$$

In the orthogonal rheometer, the two components of the tangential force on one of the plates are measured. Writing X and Y for the components in the x and y directions, respectively, we have for a plate of radius R

$$X = \int_0^R \int_0^{2\pi} [p_{(rz)}\cos\theta - p_{(\theta z)}\sin\theta]\, r\, d\theta\, dr, \tag{47}$$

$$Y = \int_0^R \int_0^{2\pi} [p_{(zr)}\sin\theta + p_{(\theta z)}\cos\theta]\, r\, d\theta\, dr. \tag{48}$$

Substituting (45)–(46) into (47)–(48), we obtain for the plate at $z = h$ (say)

$$X - iY = \frac{a\Omega\pi R^2 \eta^*\alpha(1 + \cosh\alpha h)}{2\sinh\alpha h}. \tag{49}$$

If αh is small, an approximate formula can be obtained simply by expanding the hyperbolic functions in powers of αh. In this way, we obtain

$$X - iY = \frac{a\pi R^2\Omega\eta^*}{h}\left[1 + \frac{\alpha^2 h^2}{12} - \frac{\alpha^4 h^4}{720}\right], \tag{50}$$

where terms of order $\alpha^6 h^6$ have been ignored. When fluid inertia is negligible, we have

$$X - iY = (a\Omega\pi R^2/h)\eta^* \tag{51}$$

or

$$X = (a\Omega\pi R^2/h)\eta', \tag{52}$$

$$Y = (a\pi R^2/h)G', \tag{53}$$

which are essentially the same as the expressions given by Bird & Harris (1968).

We see from (52) and (53) that the orthogonal rheometer can be used to determine directly the dynamic viscosity and dynamic rigidity when fluid inertia is negligible and (49) or (50) are now available to interpret experimental results in situations where fluid inertia is important.

REFERENCES

BIRD, R. B. & HARRIS, E. K. 1968 *Am. Inst. Chem. Eng. J.* **14**, 758.
BLYLER, L. L. & KURTZ, S. J. 1967 *J. Appl. Polym. Sci.* **11**, 127.
MAXWELL, B. & CHARTOFF, R. P. 1965 *Trans. Soc. Rheol.* **9**, 41.
OLDROYD, J. G. 1950 *Proc. Roy. Soc. Lond.* A **200**, 523.
WALTERS, K. 1970 *J. Fluid Mech.* **40**, 191.