§7. The equations of equilibrium for isotropic bodies

Let us now derive the equations of equilibrium for isotropic solid bodies. To do so, we substitute in the general equations (2.7)

$$\partial\sigma_{ik}/\partial x_k + \rho g_i = 0$$

the expression (5.11) for the stress tensor. We have

$$\frac{\partial\sigma_{ik}}{\partial x_k} = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}\frac{\partial u_{ll}}{\partial x_i} + \frac{E}{1+\sigma}\frac{\partial u_{ik}}{\partial x_k}.$$

Substituting

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right),$$

we obtain the equations of equilibrium in the form

$$\frac{E}{2(1+\sigma)}\frac{\partial^2 u_i}{\partial x_k{}^2} + \frac{E}{2(1+\sigma)(1-2\sigma)}\frac{\partial^2 u_l}{\partial x_i \partial x_l} + \rho g_i = 0. \quad (7.1)$$

These equations can be conveniently rewritten in vector notation. The quantities $\partial^2 u_i/\partial x_k{}^2$ are components of the vector $\triangle \mathbf{u}$, and $\partial u_l/\partial x_l \equiv \operatorname{div} \mathbf{u}$. Thus the equations of equilibrium become

$$\triangle \mathbf{u} + \frac{1}{1-2\sigma}\,\mathbf{grad}\operatorname{div}\mathbf{u} = -\rho\mathbf{g}\frac{2(1+\sigma)}{E}. \quad (7.2)$$

It is sometimes useful to transform this equation by using the vector identity $\mathbf{grad}\operatorname{div}\mathbf{u} = \triangle\mathbf{u} + \mathbf{curl}\,\mathbf{curl}\,\mathbf{u}$. Then (7.2) becomes

$$\mathbf{grad}\operatorname{div}\mathbf{u} - \frac{1-2\sigma}{2(1-\sigma)}\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = -\rho\mathbf{g}\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}. \quad (7.3)$$

We have written the equations of equilibrium for a uniform gravitational field, since this is the body force most usually encountered in the theory of elasticity. If there are other body forces, the vector $\rho\mathbf{g}$ on the right-hand side of the equation must be replaced accordingly.

A very important case is that where the deformation of the body is caused, not by body forces, but by forces applied to its surface. The equation of equilibrium then becomes

$$(1-2\sigma)\triangle\mathbf{u} + \mathbf{grad}\operatorname{div}\mathbf{u} = 0 \quad (7.4)$$

or

$$2(1-\sigma)\,\mathbf{grad}\operatorname{div}\mathbf{u} - (1-2\sigma)\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = 0. \quad (7.5)$$

The external forces appear in the solution only through the boundary conditions.

Taking the divergence of equation (7.4) and using the identity

$$\operatorname{div}\mathbf{grad} \equiv \triangle,$$

we find

$$\triangle \operatorname{div}\mathbf{u} = 0, \quad (7.6)$$

i.e. $\operatorname{div}\mathbf{u}$ (which determines the volume change due to the deformation) is a harmonic function. Taking the Laplacian of equation (7.4), we then obtain

$$\triangle\triangle\mathbf{u} = 0, \tag{7.7}$$

i.e. in equilibrium the displacement vector satisfies the *biharmonic equation*. These results remain valid in a uniform gravitational field (since the right-hand side of equation (7.2) gives zero on differentiation), but not in the general case of external forces which vary through the body.

The fact that the displacement vector satisfies the biharmonic equation does not, of course, mean that the general integral of the equations of equilibrium (in the absence of body forces) is an arbitrary biharmonic vector; it must be remembered that the function $\mathbf{u}(x, y, z)$ also satisfies the lower-order differential equation (7.4). It is possible, however, to express the general integral of the equations of equilibrium in terms of the derivatives of an arbitrary biharmonic vector (see Problem 10).

If the body is non-uniformly heated, an additional term appears in the equation of equilibrium. The stress tensor must include the term

$$-K\alpha(T - T_0)\delta_{ik}$$

(see (6.2)), and $\partial\sigma_{ik}/\partial x_k$ accordingly contains a term

$$-K\alpha\partial T/\partial x_i = -[E\alpha/3(1-2\sigma)]\partial T/\partial x_i.$$

The equation of equilibrium thus takes the form

$$\frac{3(1-\sigma)}{1+\sigma}\,\mathbf{grad}\operatorname{div}\mathbf{u} - \frac{3(1-2\sigma)}{2(1+\sigma)}\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \alpha\,\mathbf{grad}\,T. \tag{7.8}$$

Let us consider the particular case of a *plane deformation*, in which one component of the displacement vector (u_z) is zero throughout the body, while the components u_x, u_y depend only on x and y. The components u_{zz}, u_{xz}, u_{yz} of the strain tensor then vanish identically, and therefore so do the components σ_{xz}, σ_{yz} of the stress tensor (but not the longitudinal stress σ_{zz}, the existence of which is implied by the constancy of the length of the body in the z-direction).†

Since all quantities are independent of the coordinate z, the equations of equilibrium (in the absence of external body forces) $\partial\sigma_{ik}/\partial x_k = 0$ reduce in this case to two equations:

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0, \quad \frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = 0. \tag{7.9}$$

The most general functions σ_{xx}, σ_{xy}, σ_{yy} satisfying these equations are of the form

$$\sigma_{xx} = \partial^2\chi/\partial y^2, \qquad \sigma_{xy} = -\partial^2\chi/\partial x\partial y, \qquad \sigma_{yy} = \partial^2\chi/\partial x^2, \tag{7.10}$$

where χ is an arbitrary function of x and y. It is easy to obtain an equation which must be satisfied by this function. Such an equation must exist, since the three quantities σ_{xx}, σ_{xy}, σ_{yy} can be expressed in terms of the two quantities u_x, u_y, and are therefore not independent. Using formulae (5.13), we find, for a plane deformation,

$$\sigma_{xx} + \sigma_{yy} = E(u_{xx} + u_{yy})/(1+\sigma)(1-2\sigma).$$

† The use of the theory of functions of a complex variable provides very powerful methods of solving plane problems in the theory of elasticity. See N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, 2nd English ed., P. Noordhoff, Groningen 1963.

But

$$\sigma_{xx} + \sigma_{yy} = \triangle\chi, \qquad u_{xx} + u_{yy} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \equiv \operatorname{div} \mathbf{u},$$

and, since by (7.6) div **u** is harmonic, we conclude that the function χ satisfies the equation

$$\triangle\triangle\chi = 0, \tag{7.11}$$

i.e. it is biharmonic. This function is called the *stress function*. When the plane problem has been solved and the function χ is known, the longitudinal stress σ_{zz} is determined at once from the formula

$$\sigma_{zz} = \sigma E(u_{xx} + u_{yy})/(1+\sigma)(1-2\sigma) = \sigma(\sigma_{xx} + \sigma_{yy}),$$

or

$$\sigma_{zz} = \sigma\triangle\chi. \tag{7.12}$$

PROBLEMS

PROBLEM 1. Determine the deformation of a long rod (with length l) standing vertically in a gravitational field.

SOLUTION. We take the z-axis along the axis of the rod, and the xy-plane in the plane of its lower end. The equations of equilibrium are $\partial\sigma_{xi}/\partial x_i = \partial\sigma_{yi}/\partial x_i = 0$, $\partial\sigma_{zi}/\partial x_i = \rho g$. On the sides of the rod all the components σ_{ik} except σ_{zz} must vanish, and on the upper end ($z = l$) $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$. The solution of the equations of equilibrium satisfying these conditions is $\sigma_{zz} = -\rho g(l-z)$, with all other σ_{ik} zero. From σ_{ik} we find u_{ik} to be $u_{xx} = u_{yy} = \sigma\rho g(l-z)/E$, $u_{zz} = -\rho g(l-z)/E$, $u_{xy} = u_{xz} = u_{yz} = 0$, and hence by integration we have the components of the displacement vector, $u_x = \sigma\rho g(l-z)x/E$, $u_y = \sigma\rho g(l-z)y/E$, $u_z = -(\rho g/2E)\{l^2 - (l-z)^2 - \sigma(x^2+y^2)\}$. The expression for u_z satisfies the boundary condition $u_z = 0$ only at one point on the lower end of the rod. Hence the solution obtained is not valid near the lower end.

PROBLEM 2. Determine the deformation of a hollow sphere (with external and internal radii R_2 and R_1) with a pressure p_1 inside and p_2 outside.

SOLUTION. We use spherical polar coordinates, with the origin at the centre of the sphere. The displacement vector **u** is everywhere radial, and is a function of r alone. Hence curl **u** = 0, and equation (7.5) becomes grad div **u** = 0. Hence

$$\operatorname{div} \mathbf{u} = \frac{1}{r^2}\frac{d(r^2 u)}{dr} = \text{constant} \equiv 3a,$$

or $u = ar + b/r^2$. The components of the strain tensor are (see formulae (1.7)) $u_{rr} = a - 2b/r^3$, $u_{\theta\theta} = u_{\phi\phi} = a + b/r^3$. The radial stress is

$$\sigma_{rr} = \frac{E}{(1+\sigma)(1-2\sigma)}\{(1-\sigma)u_{rr} + 2\sigma u_{\theta\theta}\} = \frac{E}{1-2\sigma}a - \frac{2E}{1+\sigma}\frac{b}{r^3}.$$

The constants a and b are determined from the boundary conditions: $\sigma_{rr} = -p_1$ at $r = R_1$, and $\sigma_{rr} = -p_2$ at $r = R_2$. Hence we find

$$a = \frac{p_1R_1^3 - p_2R_2^3}{R_2^3 - R_1^3}\cdot\frac{1-2\sigma}{E}, \qquad b = \frac{R_1^3R_2^3(p_1 - p_2)}{R_2^3 - R_1^3}\cdot\frac{1+\sigma}{2E}.$$

For example, the stress distribution in a spherical shell with a pressure $p_1 = p$ inside and $p_2 = 0$ outside is given by

$$\sigma_{rr} = \frac{pR_1^3}{R_2^3 - R_1^3}\left(1 - \frac{R_2^3}{r^3}\right), \qquad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{pR_1^3}{R_2^3 - R_1^3}\left(1 + \frac{R_2^3}{2r^3}\right).$$

For a thin spherical shell with thickness $h = R_2 - R_1 \ll R$ we have approximately

$$u = pR^2(1-\sigma)/2Eh, \qquad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \tfrac{1}{2}pR/h, \qquad \bar{\sigma}_{rr} = \tfrac{1}{2}p,$$

where $\bar{\sigma}_{rr}$ is the mean value of the radial stress over the thickness of the shell.

The stress distribution in an infinite elastic medium with a spherical cavity (with radius R) subjected to hydrostatic compression is obtained by putting $R_1 = R$, $R_2 = \infty$, $p_1 = 0$, $p_2 = p$:

$$\sigma_{rr} = -p\left(1 - \frac{R^3}{r^3}\right), \qquad \sigma_{\theta\theta} = \sigma_{\phi\phi} = -p\left(1 + \frac{R^3}{2r^3}\right).$$

At the surface of the cavity the tangential stresses $\sigma_{\theta\theta} = \sigma_{\phi\phi} = -3p/2$, i.e. they exceed the pressure at infinity.

PROBLEM 3. Determine the deformation of a solid sphere (with radius R) in its own gravitational field.

SOLUTION. The force of gravity on unit mass in a spherical body is $-g\mathbf{r}/R$. Substituting this expression in place of $\mathbf{g}$ in equation (7.3), we obtain the following equation for the radial displacement:

$$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\frac{d}{dr}\left(\frac{1}{r^2}\frac{d(r^2u)}{dr}\right) = \rho g\frac{r}{R}.$$

The solution finite for $r = 0$ which satisfies the condition $\sigma_{rr} = 0$ for $r = R$ is

$$u = -\frac{g\rho R(1-2\sigma)(1+\sigma)}{10E(1-\sigma)}r\left(\frac{3-\sigma}{1+\sigma} - \frac{r^2}{R^2}\right).$$

It should be noticed that the substance is compressed ($u_{rr} < 0$) inside a spherical surface of radius $R\sqrt{\{(3-\sigma)/3(1+\sigma)\}}$ and stretched outside it ($u_{rr} > 0$). The pressure at the centre of the sphere is $(3-\sigma)g\rho R/10(1-\sigma)$.

PROBLEM 4. Determine the deformation of a cylindrical pipe (with external and internal radii R_2 and R_1), with a pressure p inside and no pressure outside.†

SOLUTION. We use cylindrical polar coordinates, with the z-axis along the axis of the pipe. When the pressure is uniform along the pipe, the deformation is a purely radial displacement $u_r = u(r)$. Similarly to Problem 2, we have

$$\operatorname{div}\mathbf{u} = \frac{1}{r}\frac{d(ru)}{dr} = \text{constant} \equiv 2a.$$

Hence $u = ar + b/r$. The non-zero components of the strain tensor are (see formulae (1.8)) $u_{rr} = du/dr = a - b/r^2$, $u_{\phi\phi} = u/r = a + b/r^2$. From the conditions $\sigma_{rr} = 0$ at $r = R_2$, and $\sigma_{rr} = -p$ at $r = R_1$, we find

$$a = \frac{pR_1^2}{R_2^2 - R_1^2}\cdot\frac{(1+\sigma)(1-2\sigma)}{E}, \qquad b = \frac{pR_1^2R_2^2}{R_2^2 - R_1^2}\cdot\frac{1+\sigma}{E}.$$

The stress distribution is given by the formulae

$$\sigma_{rr} = \frac{pR_1^2}{R_2^2 - R_1^2}\left(1 - \frac{R_2^2}{r^2}\right), \qquad \sigma_{\phi\phi} = \frac{pR_1^2}{R_2^2 - R_1^2}\left(1 + \frac{R_2^2}{r^2}\right),$$

$$\sigma_{zz} = 2p\sigma R_1^2/(R_2^2 - R_1^2).$$

PROBLEM 5. Determine the deformation of a cylinder rotating uniformly about its axis.

SOLUTION. Replacing the gravitational force in (7.3) by the centrifugal force $\rho\Omega^2 r$ (where Ω is the angular velocity), we have in cylindrical polar coordinates the following equation for the displacement $u_r = u(r)$:

$$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\frac{d}{dr}\left(\frac{1}{r}\frac{d(ru)}{dr}\right) = -\rho\Omega^2 r.$$

The solution which is finite for $r = 0$ and satisfies the condition $\sigma_{rr} = 0$ for $r = R$ is

$$u = \frac{\rho\Omega^2(1+\sigma)(1-2\sigma)}{8E(1-\sigma)}r[(3-2\sigma)R^2 - r^2].$$

PROBLEM 6. Determine the deformation of a non-uniformly heated sphere with a spherically symmetrical temperature distribution.

† In Problems 4, 5 and 7 it is assumed that the length of the cylinder is maintained constant, so that there is no longitudinal deformation.