1. Let $\mathbb{A} \in \mathbb{R}^{3\times 3}$ be an invertible matrix. Show that

$$\operatorname{Tr}(\operatorname{cof}\mathbb{A}) = \frac{1}{2}\left((\operatorname{Tr}\mathbb{A})^2 - \operatorname{Tr}\left(\mathbb{A}^2\right)\right).$$

   (We have in fact discussed this at the lecture, just do it carefully. I recall that the most convenient way for proving the formula is based on Schur decomposition. Schur decomposition is also know as Schur triangulisation theorem, and you should be familiar with it from the basic course on linear algebra.)

2. Let $\mathbb{A} \in \mathbb{R}^{3\times 3}$ be an invertible matrix and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be arbitrary fixed vectors in $\mathbb{R}^3$ such that $\boldsymbol{v} \bullet \mathbb{A}^{-1}\boldsymbol{u} \neq -1$. Show that

$$(\mathbb{A} + \boldsymbol{u}\otimes\boldsymbol{v})^{-1} = \mathbb{A}^{-1} - \frac{1}{1 + \boldsymbol{v}\bullet\mathbb{A}^{-1}\boldsymbol{u}}\left(\mathbb{A}^{-1}\boldsymbol{u}\right)\otimes\left(\mathbb{A}^{-\top}\boldsymbol{v}\right).$$

   The formula is usually referred to as the Sherman–Morrison formula.

3. We already know that

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km},$$

   and let us further assume that we also know that

$$\epsilon_{ijk}\epsilon_{lmn} = \det\begin{bmatrix}\delta_{il} & \delta_{im} & \delta_{in}\\ \delta_{jl} & \delta_{jm} & \delta_{jn}\\ \delta_{kl} & \delta_{km} & \delta_{kn}\end{bmatrix}.$$

   - Show that

$$\epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn}.$$

   - Show that

$$\epsilon_{ijk}\delta_{lm} = \epsilon_{jkm}\delta_{il} + \epsilon_{kim}\delta_{jl} + \epsilon_{ijm}\delta_{kl}.$$

4. [Optional] Use formula $\epsilon_{ijk}\delta_{lm} = \epsilon_{jkm}\delta_{il} + \epsilon_{kim}\delta_{jl} + \epsilon_{ijm}\delta_{kl}$, and show that for any triple of non-coplanar vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ one has

$$\operatorname{Tr}\mathbb{A} = \frac{\mathbb{A}\boldsymbol{u}\bullet(\boldsymbol{v}\times\boldsymbol{w}) + \boldsymbol{u}\bullet(\mathbb{A}\boldsymbol{v}\times\boldsymbol{w}) + \boldsymbol{u}\bullet(\boldsymbol{v}\times\mathbb{A}\boldsymbol{w})}{\boldsymbol{u}\bullet(\boldsymbol{v}\times\boldsymbol{w})}.$$

5. [Optional] Do you think that it is true that if you know the spectrum (all eigenvalues) of a matrix, then you know everything about the matrix? In particular, is it always possible to express the Frobenius norm of a matrix, $|\mathbb{A}| =_{\text{def}} \left(\operatorname{Tr}\left(\mathbb{A}\mathbb{A}^\top\right)\right)^{\frac{1}{2}}$, as a function of eigenvalues?