1. Show that

$$\frac{\partial^2 \mathrm{I}_1(\mathbb{A})}{\partial \mathbb{A}^2}[\mathbb{B}, \mathbb{C}] = 0,$$
$$\frac{\partial^2 \mathrm{I}_2(\mathbb{A})}{\partial \mathbb{A}^2}[\mathbb{B}, \mathbb{C}] = (\operatorname{Tr} \mathbb{C})(\operatorname{Tr} \mathbb{B}) - \operatorname{Tr}(\mathbb{C}\mathbb{B}),$$
$$\frac{\partial^2 \mathrm{I}_3(\mathbb{A})}{\partial \mathbb{A}^2}[\mathbb{B}, \mathbb{C}] = (\det \mathbb{A})\left(\operatorname{Tr}\left(\mathbb{A}^{-1}\mathbb{B}\right)\operatorname{Tr}\left(\mathbb{A}^{-1}\mathbb{C}\right) - \operatorname{Tr}\left(\mathbb{A}^{-1}\mathbb{B}\mathbb{A}^{-1}\mathbb{C}\right)\right),$$

where $\mathrm{I}_1(\mathbb{A})$, $\mathrm{I}_2(\mathbb{A})$ and $\mathrm{I}_3(\mathbb{A})$ denote the principal invariants of matrix $\mathbb{A}$, that is

$$\mathrm{I}_1(\mathbb{A}) =_{\mathrm{def}} \operatorname{Tr} \mathbb{A},$$
$$\mathrm{I}_2(\mathbb{A}) =_{\mathrm{def}} \frac{1}{2}\left((\operatorname{Tr} \mathbb{A})^2 - \operatorname{Tr}\left(\mathbb{A}^2\right)\right),$$
$$\mathrm{I}_3(\mathbb{A}) =_{\mathrm{def}} \det \mathbb{A}.$$

2. [Optional] Let $\mathbb{U}$ be the symmetric positive definite matrix from the polar decomposition theorem, that is $\mathbb{U} = (\mathbb{F}^\top \mathbb{F})^{\frac{1}{2}}$, where $\mathbb{F}$ is an invertible matrix with $\det \mathbb{F} > 0$. Show that

$$\frac{\partial \mathbb{U}}{\partial \mathbb{F}}[\mathbb{B}] = \int_{s=0}^{+\infty} \mathrm{e}^{-\mathbb{U}s}\left(\mathbb{B}^\top \mathbb{F} + \mathbb{F}^\top \mathbb{B}\right)\mathrm{e}^{-\mathbb{U}s}\,\mathrm{d}s.$$

Please note that $\frac{\partial \mathfrak{f}(\mathbb{A})}{\partial \mathbb{A}}[\mathbb{B}]$ is just another notation for Gâteaux derivative, that is

$$\frac{\partial \mathfrak{f}(\mathbb{A})}{\partial \mathbb{A}}[\mathbb{B}] =_{\mathrm{def}} \mathrm{D}_{\mathbb{A}}\mathfrak{f}(\mathbb{A})[\mathbb{B}].$$