Must Elastic Materials be Hyperelastic?

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Abstract: Rivlin's observation that elastic stress response that does not derive from a strain energy (Cauchy elasticity) admits negative work in closed cycles of deformations is demonstrated for specific isotropic stress responses and for a simple closed cycle of deformation. It is concluded that elastic materials must be hyperelastic.

Key Words: finite elasticity, Cauchy elasticity, hyperelasticity, strain energy

1. INTRODUCTION

In the finite strain theory of elasticity, the material response is typically modeled in one of two ways. In the first model (called *elasticity* or *Cauchy elasticity*) the stress response is assumed. In the second model (called *hyperelasticity* or *Green elasticity*) a strain energy is assumed and the stress response is derived. These two formulations are equivalent if the stress response function in the first model is assumed, at least tacitly, to be derivable from a strain energy. Then the adoption of one approach or the other is a matter of taste or convenience, although failure to make explicit the existence of a strain energy will lead to some loss of simplification associated with, for example, the availability of the energy momentum tensor or the inverse deformation theorem, or the lack of symmetry of the acoustic tensor. The two approaches become significantly different, however, if the first theory is presented as a physically viable theory that is broader than hyperelasticity, which it includes as a special case. The different terminology suggests that there is such a difference and this is reflected in the presentations of elasticity and hyperelasticity in textbooks and monographs.

During a discussion at the International Symposium on Second-order Effects at Haifa in 1962 [1], Rivlin pointed out that an elastic material that is not hyperelastic would allow for non-zero work in closed cycles of deformation. This, in turn, would imply that the work done in completing the cycle in the appropriate sense would be negative, since changing the sense in which the cycle is traversed changes the sign of the work done. Negative work in a closed cycle of deformation means that the energy released during unloading *exceeds* the work done during loading, so that the material becomes a source of energy. Indeed, since the cycle may be repeated indefinitely, the material is a source of unlimited energy. Truesdell dismissed

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Rivlin's claim as a "silly shibboleth" but he did not refute it and a general agreement on this obviously important issue has not been reached since then. Truesdell [2] identified the problem of ensuring physically reasonable behavior as the "Hauptproblem" of elasticity.

The specialization of a thermodynamical theory of elasticity to a purely mechanical theory (for either isothermal or isentropic response) leads to a strain energy function. Proponents of this theory maintain that an elasticity theory without a strain energy is not physically reasonable for the reason stated above. Proponents of Cauchy elasticity argue that the assumption of a strain energy is overly restrictive and prohibits a modeling of the behavior of inelastic materials under certain circumstances, although it is not clear what these circumstances might be. Presumably, they do not include closed cycles of deformation.

In the present note, Rivlin's idea is made concrete by choosing specific "non-hyperelastic" stress response laws and by calculating the work done in a simple closed cycle of irrotational biaxial homogeneous plane strain of a cube. The stress responses considered include a simple extension of the classical Hooke's law of isotropic linear elasticity to the finite strain region by replacing the infinitesimal strain tensor by the logarithmic strain tensor, as well as two non-hyperelastic perturbations of the general isotropic hyperelastic response. For each case, it is shown that the work in the closed cycle is positive or negative depending on the order in which the two stretches are applied and released, i.e., on whether the greater or the lesser stretch is effected first.

It should be emphasized that these results merely demonstrate Rivlin's statement for specific materials and cycles. Indeed, Rivlin's statement has not been challenged; it is its implications that are questioned. Hopefully, these simple concrete examples will help toward a resolution of this important question of the viability of Cauchy elasticity.

2. ISOTROPIC ELASTIC AND HYPERELASTIC RESPONSE

A deformation

$$\mathbf{x} = \mathbf{x}(\mathbf{X}) \tag{2.1}$$

takes a typical particle from a place $\mathbf{X}$ in the undeformed configuration to a place $\mathbf{x}$ in the deformed configuration. The deformation gradient tensor $\mathbf{F} = \text{Grad}\,\mathbf{x}$ describes the local deformation. Since the Jacobian $\det \mathbf{F}$ measures the local change in volume, we have $0 < \det \mathbf{F} < \infty$. Accordingly, $\mathbf{F}$ admits polar decompositions

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{2.2}$$

where the rotation $\mathbf{R}$ is proper orthogonal and the stretch tensors $\mathbf{U}$ and $\mathbf{V}$ are symmetric and positive-definite. The tensors $\mathbf{F}$, $\mathbf{R}$, $\mathbf{U}$ and $\mathbf{V}$ admit representations

$$\mathbf{F} = \lambda_i \mathbf{v}^i \otimes \mathbf{u}^i; \quad \mathbf{R} = \mathbf{v}^i \otimes \mathbf{u}^i; \quad \mathbf{U} = \lambda_i \mathbf{u}^i \otimes \mathbf{u}^i; \quad \mathbf{V} = \lambda_i \mathbf{v}^i \otimes \mathbf{v}^i. \tag{2.3}$$

The right orthonormal triads $(\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3)$ and $(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3)$ are the right and left principal directions of stretch, the corresponding eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ are the principal stretches, and the repeated index in (2.3) is summed over the range $i = 1, 2, 3$.

For an isotropic elastic solid without internal constraint, the response law for the Piola stress has the form

$$\mathbf{P} = \mathbf{R}\boldsymbol{\Phi}(\mathbf{U}), \tag{2.4}$$

with

$$\boldsymbol{\Phi}(\mathbf{Q}\mathbf{U}\mathbf{Q}^{\mathrm{T}}) = \mathbf{Q}\boldsymbol{\Phi}(\mathbf{U})\mathbf{Q}^{\mathrm{T}} \tag{2.5}$$

for all proper orthogonal tensors $\mathbf{Q}$. Correspondingly, the Cauchy stress response is

$$\mathbf{T} = \mathbf{R}\boldsymbol{\Psi}(\mathbf{U})\mathbf{R}^{\mathrm{T}} = \boldsymbol{\Psi}(\mathbf{V}), \tag{2.6}$$

with

$$\boldsymbol{\Psi}(\mathbf{U}) = 1/\det \mathbf{U}\boldsymbol{\Phi}(\mathbf{U})\mathbf{U}. \tag{2.7}$$

The Piola and Cauchy stress tensors admit representations

$$\mathbf{P} = p_i \mathbf{v}^i \otimes \mathbf{u}^i; \quad \mathbf{T} = t_i \mathbf{v}^i \otimes \mathbf{v}^i \tag{2.8}$$

and the principal forces p_i and principal stresses t_i are related through

$$t_i = p_i/\lambda_j\lambda_k \quad (i \neq j \neq k \neq i). \tag{2.9}$$

It follows from (2.4), (2.5) and (2.8) that the dependence of the principal forces on the principal stretches has the form

$$p_i = p(\lambda_i, \lambda_j, \lambda_k) \quad (i \neq j \neq k \neq i), \tag{2.10}$$

where the function p is symmetric in its second and third arguments.

The work done by the contact forces, per unit initial volume, in carrying the material from a stretch state Λ: $(\lambda_1, \lambda_2, \lambda_3)$ to a stretch state Γ: $(\gamma_1, \gamma_2, \gamma_3)$ is a line integral

$$W(\Lambda, \Gamma) = \int_{\Lambda}^{\Gamma} p_i \mathrm{d}\lambda_i. \tag{2.11}$$

The strain energy W per unit volume (if there is one) may be written as a function of the principal stretches

$$W = w(\lambda_1, \lambda_2, \lambda_3) \tag{2.12}$$

that is fully symmetric in its arguments. Alternatively, the strain energy may be written as a function of the principal invariants of $\mathbf{U}$, or of $\mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, but such representations are not useful for the present purpose. The principal forces and stresses are derivable from the strain energy function as

$$p_i = \partial w/\partial \lambda_i; \quad t_i = 1/\lambda_j \lambda_k \partial w/\partial \lambda_i \quad (i \neq j \neq k \neq i). \tag{2.13}$$

It follows from $(2.13)_1$ that the integrand in (2.11) is an exact differential, so that the work

$$W(\Lambda;\Gamma) = \int_\Lambda^\Gamma \mathrm{d}w = w(\gamma_1, \gamma_2, \gamma_3) - w(\lambda_1, \lambda_2, \lambda_3) \tag{2.14}$$

is path independent. In particular, the work done in taking the material through a closed cycle of deformation is zero:

$$W(\Lambda;\Lambda) = \int \mathrm{d}w = 0. \tag{2.15}$$

It is evident from $(2.13)_1$ that the condition on the principal force response law (2.10) to ensure the existence of a strain energy function is

$$\partial p_i/\partial \lambda_j = \partial p_j/\partial \lambda_i. \tag{2.16}$$

3. A SPECIFIC ELASTIC RESPONSE LAW

Consider the stress response law (2.4) with

$$\Phi(\mathbf{U}) = 2\mu \ln \mathbf{U} + \beta(\mathrm{tr}\ \ln \mathbf{U})\mathbf{1} \quad (\mu > 0, \ \ \beta > 0), \tag{3.1}$$

where $\mathbf{1}$ is the unit tensor. This seems quite appealing; it extends the classical isotropic stress-strain law of linear elasticity, with Lamé constants μ and β, to the finite strain range by replacing the infinitesimal strain tensor by the logarithmic strain tensor. The principal forces are linear in the principal logarithmic strains

$$p_i = 2\mu\varepsilon_i + \beta(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \quad (\varepsilon_i = \ln \lambda_i). \tag{3.2}$$

It is easy to verify that the Baker–Ericksen inequalities [3], which require that the ordering of the principal stresses t_i and principal stretches λ_i be the same,

$$(t_i - t_j)(\lambda_i - \lambda_j) > 0 \quad \text{for} \quad \lambda_i \neq \lambda_j. \tag{3.3}$$

are met in the tensile region $\lambda_i \geq 1$.

It is convenient to introduce the modulus $\alpha = 2\mu + \beta$. Then the principal forces are

$$p_i = \alpha \ln\lambda_i + \beta \ln\lambda_j\lambda_k \quad (i \neq j \neq k \neq i). \tag{3.4}$$

It follows from (3.2) that

$$\partial p_i/\partial\lambda_j = \beta/\lambda_j \quad (i \neq j), \tag{3.5}$$

which does not pass the test (2.16), since $\beta \neq 0$. So, the material with stress response described by (2.4) and (3.1) is elastic but it is not hyperelastic. It is evident that the problem (if it is a problem?) is with the β-term in (3.5); the α-term derives from a strain energy

$$W = \alpha\{\lambda_1\ln\lambda_1 + \lambda_2\ln\lambda_2 + \lambda_3\ln\lambda_3 - (\lambda_1 + \lambda_2 + \lambda_3 - 3)\}. \tag{3.6}$$

4. A SPECIFIC CLOSED CYCLE OF DEFORMATION

Consider a homogeneous, irrotational plane strain in the tensile region, i.e.,

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = X_3 \quad (\lambda_1 \geq 1, \ \ \lambda_2 \geq 1), \tag{4.1}$$

where (X_1, X_2, X_3) and (x_1, x_2, x_3) are rectangular Cartesian coordinates. The corresponding principal forces are (from (3.4))

$$p_1 = \alpha\ln\lambda_1 + \beta\ln\lambda_2, \quad p_2 = \alpha\ln\lambda_2 + \beta\ln\lambda_1, \quad p_3 = \beta\ln\lambda_1\lambda_2. \tag{4.2}$$

Consider the following closed cycle of deformation of a unit cube of material:

1. Stretch to $\lambda_1 = a$ (> 1) holding λ_2 constant at $\lambda_2 = 1$ (ABCD to AEFD in Figure 1).
2. Stretch to $\lambda_2 = b$ (> 1) holding λ_1 constant at $\lambda_1 =$ a (AEFD to AEGH).
3. Relax to $\lambda_1 = 1$ holding λ_2 constant at $\lambda_2 =$ b (AEGH to ABIH).
4. Relax to $\lambda_2 = 1$ holding λ_1 constant at $\lambda_1 = 1$ (ABIH to ABCD).

The path in the λ_1, λ_2-plane is PQRSP in Figure 2.

Each phase of this deformation is effected by principal forces p$_i$ uniformly distributed over the six faces of the cube and given by (4.2). The work done by these principal forces in carrying out this deformation is

$$W = \int (p_1 \mathrm{d}\lambda_1 + p_2 \mathrm{d}\lambda_2 + p_3 \mathrm{d}\lambda_3). \tag{4.3}$$

In accordance with (2.15), the α terms in (4.2) do not contribute, since they are derivable from a strain energy. The principal force p_3 does no work since $\lambda_3 = 1$ throughout. Thus, the total work is the "non-hyperelastic" contribution

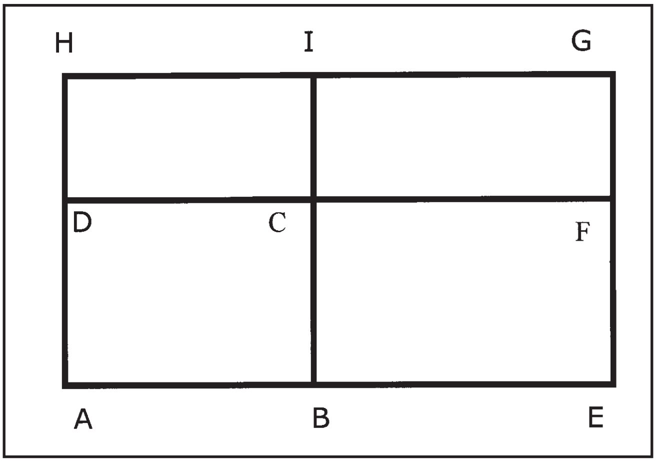

Figure 1.

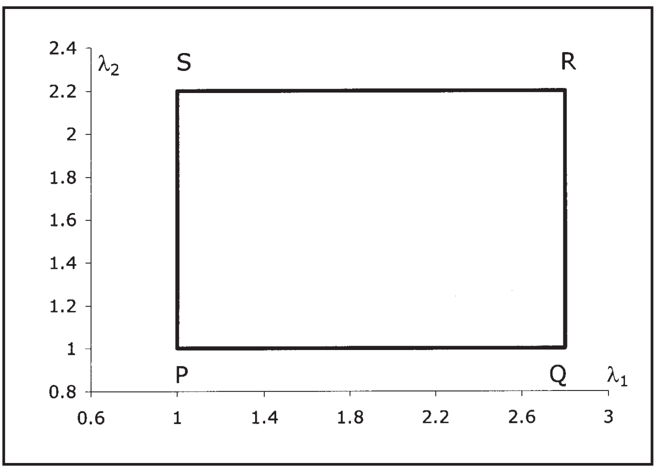

Figure 2.

$$W = \beta \int (\ln\lambda_2 d\lambda_1 + \ln\lambda_1 d\lambda_2). \tag{4.4}$$

There is no contribution during the first stretching phase PQ ($\lambda_2 = 1$) or the second relaxation phase SP ($\lambda_1 = 1$). The work done along QRS is

$$W = \beta\{(b-1)\ln a - (a-1)\ln b\}, \tag{4.5}$$

which is nonzero if $a \neq b$ and is negative if $a > b$, i.e., if the greater stretch is effected first.

Of course, this problem is not peculiar to the stress response (3.4). As Rivlin [1] pointed out, the problem is common to all elastic materials that are not hyperelastic. Non-zero work in closed cycles of deformation implies negative work if the stretch path is described in the appropriate sense and so all non-hyperelastic materials will manifest this in some closed cycles. A non-hyperelastic perturbation of a general hyperelastic response

$$p_i = \partial w/\partial \lambda_i + \beta \ln \lambda_j \lambda_k \quad (i \neq j \neq k \neq i), \tag{4.6}$$

where $w(\lambda_1, \lambda_2, \lambda_3)$ is any isotropic strain energy function, will again yield the work (4.5). There is nothing special about this particular perturbation; The response law

$$p_i = \partial w/\partial \lambda_i + \beta/\lambda_j \lambda_k \quad (i \neq j \neq k \neq i), \tag{4.7}$$

does not derive from a strain energy and gives the total work

$$W = \beta(a-b)(a-1)(b-1)/ab, \tag{4.8}$$

for this same cycle of deformation. This is negative if $b > a$, i.e., if the lesser stretch is effected first.

5. DISCUSSION

It is probably not worthwhile to attempt to chronicle the ongoing debate over the validity of Cauchy elasticity. Indeed, arguments on both sides of the issue (including the present paper) have added little of substance to Rivlin's initial assertion and Truesdell's dismissive response.

It is clear that the issue is not yet resolved. Modern texts and monographs on nonlinear elasticity typically present Cauchy elasticity as a viable general theory and introduce hyperelasticity (sometimes much later in the text) as a special case with some interesting implications such as the symmetry of the acoustic tensor. The fact that the existence of a strain energy is necessary to avoid negative work in closed cycles of deformation is rarely addressed.

A recent paper by Batra [4], for instance, presents and examines four linear stress response relations similar to (3.1) but with different measures of stress and strain. It is a simple matter to write the corresponding principal force response relations and to see that two of them meet the hyperelasticity condition (2.16) and the other two do not and so will exhibit behavior similar to that described in Section 3. The point at issue here is that all four are accorded equal status. Indeed, Batra says "Each of the four constitutive relations (1)–(4)... is consistent with the principles of Continuum Mechanics" and then cites Truesdell and Noll [5]. Unfortunately, he is quite correct, since these widely accepted principles do not require that elastic materials should not release unlimited energy during multi-parameter cyclical processes in which loads are applied *and removed*. Indeed, Truesdell and Noll [5, #92] state that "at present, on the basis of solutions of special problems, no straightforward test for the existence of a strain-energy is known". The solution described in Section 3 surely suggests such a test: Carry out the biaxial stretch test described, effecting the larger stretch first. If the material releases energy in this process, it is not hyperelastic. If the material absorbs energy in the process, carry out the test again, effecting the smaller stretch first. If the material releases energy in this process, it is not hyperelastic. Any sensible theory of elasticity should certainly encompass loading and unloading response and should preclude such physically unacceptable (albeit economically desirable!) energy release, so we conclude that elastic materials must be hyperelastic.

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