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Oscillatory Shearing of Nonlinearly Elastic Solids

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1. Introduction

Most of the available exact solutions in finite elasticity pertain to incompressible materials. These include the so-called controllable or universal deformations, the associated quasi-equilibrated motions (including radial oscillations of hollow cylinders and spheres), and some propagating wave solutions (see, for example, Truesdell and Noll [1]). There are very few exact solutions available for compressible materials, other than those involving homogeneous deformations. One exception is the solution [2] for a transverse circularly-polarized harmonic plane progressive wave, which has the unusual feature that the strain invariants associated with the nonhomogeneous motion are constant in space and time. A somewhat similar solution involving a spatially nonuniform oscillatory shearing motion, for which the strain invariants are spatially uniform but time-dependent, is presented here. Such a motion can be sustained in a compressible or incompressible homogeneous isotropic elastic solid without applying a body force. The spatial geometry is the same for all materials and the nature of the time-dependence, for a particular material, is determined by the generalized shear modulus. The motion is one which can be sustained in a layer by subjecting its two faces to rectilinear motions in different directions. A more general solution is discussed briefly, and it is also shown that a certain shearing motion with spatially non-uniform time-independent strain invariants can be sustained only in incompressible materials. Finally, the motions are shown to coexist with spatially uniform, time-dependent temperatures in thermoelastic solids, in the absence of body force and volume supply of heat.

2. Preliminaries

The response of a homogeneous istropic elastic solid to deformations from an undistorted reference configuration is described by a constitutive relation

$$T = \alpha 1 + \beta B + \gamma B^{-1}, \tag{2.1}$$

where T is the Cauchy stress tensor, 1 is the unit tensor and B is the left Cauchy-Green deformation tensor, defined by

$$B = FF^T, \tag{2.2}$$

F being the deformation gradient tensor. The scalars α, β and γ are functions of the principal invariants of B, which are denoted by I_B, II_B, III_B.

For a homogeneous simple shear

$$x = X + KZ, \qquad y = Y, \qquad z = Z, \tag{2.3}$$

where x, y, z and X, Y, Z denote the rectangular Cartesian coordinates of a particle in the deformed configuration, and in the reference configuration, respectively, the strain invariants are

$$\mathrm{I}_B = \mathrm{II}_B = 3 + K^2, \qquad \mathrm{III}_B = 1. \tag{2.4}$$

The nonzero stress components are

$$\begin{aligned} &T_{xx} = \alpha + \gamma + (1 + K^2)\beta, \quad T_{yy} = \alpha + \beta + \gamma, \quad T_{zz} = \alpha + \beta + (1 + K^2)\gamma, \\ &T_{zx} = T_{xz} = (\beta - \gamma)K, \end{aligned} \tag{2.5}$$

where α, β and γ are functions of the invariants (2.4). In particular, the ratio T_{zx}/K of the shear stress to the amount of shear is an even function of the amount of shear,

$$\hat{\mu}(K^2) = \beta(3 + K^2, 3 + K^2, 1) - \gamma(3 + K^2, 3 + K^2, 1) \tag{2.6}$$

and it is called the generalized shear modulus.

For incompressible materials, the constitutive relation (2.1) is replaced by

$$T = -p1 + \beta B + \gamma B^{-1} \tag{2.7}$$

where p is an arbitrary scalar and β and γ are functions of the strain invariants I_B and II_B (the invariant III_B has the value 1 for all possible motions). The generalized shear modulus is defined in a manner similar to (2.6).

3. An Oscillatory Shearing Motion

Consider a motion

$$x = X + f(t)\cos kZ, \qquad y = Y + f(t)\sin kZ, \qquad z = Z, \tag{3.1}$$

where x, y, z are coordinates of a particle at time t and k is a constant. A routine calculation shows that the strain invariants are

$$\mathrm{I}_B = \mathrm{II}_B = 3 + k^2 f^2, \qquad \mathrm{III}_B = 1, \tag{3.2}$$

and so depend only on t, while the stress components are independent of x and y. The stress components T_{xz}, T_{yz} and T_{zz} are

$$\begin{aligned} T_{xz} &= -\hat{\mu}(k^2 f^2) kf \sin kz, \\ T_{yz} &= \hat{\mu}(k^2 f^2) kf \cos kz, \\ T_{zz} &= \alpha + \beta + (1 + k^2 f^2)\gamma, \end{aligned} \tag{3.3}$$

so that T_{zz} is independent of z. Accordingly, the equations of motion, with no body force, are satisfied if

$$\rho \ddot{f} + \hat{\mu}(k^2f^2)k^2f = 0, \tag{3.4}$$

where the dot denotes differentiation with respect to time and the density ρ is constant, in view of $(3.2)_3$. Thus, the motion (3.1) is possible, in the absence of body force, in any compressible homogeneous isotropic elastic solid if the function $f(t)$ satisfies the differential equation (3.4). The motion depends on a limited aspect of the material response, namely, the generalized shear modulus $\hat{\mu}$, and on the initial density ρ. It is evident that such a motion is also possible in an incompressible solid.

Equation (3.4) may be integrated once to give

$$\rho \dot{f}^2 = M(k^2f_m^2) - M(k^2f^2), \tag{3.5}$$

where the function M is defined by

$$M(s) = \int_0^s \hat{\mu}(\sigma)d\sigma \tag{3.6}$$

and f_m is a constant. Thus, (3.4) generally describes a nonlinear oscillation, whose form is obtained by a further integration of (3.5), and with period τ given by

$$\tau = 2\int_{-f_m}^{f_m} \frac{df}{\dot{f}} = 2\rho^{1/2}\int_{-f_m}^{f_m} \frac{df}{\{M(k^2f_m^2) - M(k^2f^2)\}^{1/2}}. \tag{3.7}$$

The maximum amplitude f_m and the velocity $\dot{f}_0$ at the equilibrium position $f = 0$ are related through

$$\rho \dot{f}_0^2 = M(k^2f_m^2). \tag{3.8}$$

For materials whose generalized shear modulus is constant[1]) the oscillation is linear with

$$f = f_m \cos(\omega t + \phi); \qquad \omega = k(\hat{\mu}/\rho)^{1/2}. \tag{3.9}$$

It should be clear that, while the motion described by (3.1) and (3.4) represents a free oscillation of an infinite body in that no body force is needed to sustain it, it is not free for a finite or semi-infinite body since appropriate surface tractions must be applied. For a layer of material occupying the region $0 \leq Z \leq L$, for example, (3.1) describes oscillatory motion of the faces $Z = 0$ and $Z = L$ in the x-direction and in the y-direction, respectively, if

$$k = (n + \tfrac{1}{2})\pi/L \quad (n = 0, 1, 2, \ldots). \tag{3.10}$$

There is a specific amplitude function $f(t)$, found by integration of (3.5), for each mode $n = 0, 1, 2, \ldots$, and, of course, the principle of superposition does not apply. The time-dependent tractions on the faces $Z = 0$ and $Z = L$ can be read off from (3.3).

[1]) This is true, for instance, of neo-Hookean and Mooney-Rivlin materials, which are commonly used to model the nonlinear behavior of rubber.

For the reasons just stated this solution may not be a very useful one, but it is of interest as an exact solution involving finite nonhomogeneous motions of compressible or incompressible elastic solids.

4. A More General Shearing Motion

The strain invariants associated with the motion

$$
\begin{aligned}
x &= X + f(t)\cos\{kZ + \phi(t)\},\\
y &= Y + f(t)\sin\{kZ + \phi(t)\},\\
z &= Z,
\end{aligned}
\tag{4.1}
$$

are also given by (3.2). Proceeding as before leads to a coupled pair of differential equations for the functions $f(t)$ and $\phi(t)$;

$$\hat{\mu}(k^2f^2)k^2f + \rho(\ddot{f} - f\dot{\phi}^2) = 0 \quad \text{and} \quad f\ddot{\phi} + 2\dot{f}\dot{\phi} = 0. \tag{4.2}$$

The second equation implies

$$f^2\dot{\phi} = A \text{ (constant)} \tag{4.3}$$

and substituting in the first equation gives an equation for $f(t)$

$$\hat{\mu}(k^2f^2)k^2f + \rho(\ddot{f} - A^2/f^3) = 0. \tag{4.4}$$

This may be integrated once to give

$$M(k^2f^2) + \rho(\dot{f}^2 + A^2/f^2) = B \text{ (constant)} \tag{4.5}$$

and it follows that $\dot{f}$ is singular at $f = 0$ unless $A = 0$, which represents the situation discussed in the previous section. Otherwise the body does not occupy its equilibrium position at any time. If f is constant, so that ϕ is linear in t, the motion (4.1) represents the circularly-polarized plane wave discussed previously [2].

5. Motions with Time-Independent Invariants

The results of the previous sections suggest that it may also be useful to consider motions with strain invariants which are spatially nonuniform and constant in time. The motion

$$x = X + g(Z)\cos\omega t, \qquad y = Y + g(Z)\sin\omega t, \qquad z = Z, \tag{5.1}$$

for example, has invariants

$$\mathrm{I}_B = \mathrm{II}_B = 3 + g'^2, \qquad \mathrm{III}_B = 1, \tag{5.2}$$

where the prime denotes differentiation with respect to Z (or z). The stress components are independent of x and y, with T_{xz}, T_{yz} and T_{zz} given by

$$
\begin{aligned}
T_{xz} &= \mu(g'^2)g'\cos\omega t,\\
T_{yz} &= \mu(g'^2)g'\sin\omega t,\\
T_{zz} &= \alpha + \beta + (1 + g'^2)\gamma.
\end{aligned}
\tag{5.3}
$$

Since T_{zz} depends on z the motion (5.1) cannot be sustained in a compressible material without applying a body force.

For incompressible materials, however, T_{zz} is given by

$$T_{zz} = -p + \beta + (1 + g'^2)\gamma \tag{5.4}$$

and the arbitrary pressure p allows the motion if

$$\{\hat{\mu}(g'^2)g'\}' + \rho\omega^2 g = \text{constant}. \tag{5.5}$$

The more general motion described by

$$\begin{aligned} x &= X + g(Z)\cos\{\omega t + \psi(Z)\}, \\ y &= Y + g(Z)\sin\{\omega t + \psi(Z)\}, \\ z &= Z, \end{aligned} \tag{5.6}$$

has time-independent strain invariants and can be sustained in an incompressible material provided

$$(\hat{\mu}g')' - \hat{\mu}g\psi'^2 + \rho\omega^2 g = 0 \quad \text{and} \quad \hat{\mu}g^2\psi' = \text{constant}, \tag{5.7}$$

where $\hat{\mu}$ denotes $\hat{\mu}(g'^2 + g^2\psi'^2)$.

6. Thermoelastic Solids

Motions of the form (3.1) and (4.1) may also coexist with a spatially uniform temperature field, $h(t)$ say, in every homogeneous, isotropic, thermoelastic solid, in the absence of body force and heat supply.[2] The shear stress and the entropy in simple shear are functions of the amount of shear and of the temperature, the form of these functions being derived from the constitutive function which relates the Helmholtz free energy to the strain invariants and the temperature [1].

A spatially uniform temperature implies that there is no heat conduction and hence (since there is no heat supply) that the process is isentropic. Thus, the functions $f(t)$ and $h(t)$ satisfy the equations

$$\rho\ddot{f} + \hat{\mu}(k^2f^2, h)k^2 f = 0 \quad \text{and} \quad \hat{\nu}(k^2f^2, h) = \text{constant}, \tag{6.1}$$

where $\hat{\nu}$ is the response function for the specific entropy in simple shear.

A similar result applies for the motion (4.1) and, in particular, a harmonic, circularly-polarized, plane progressive wave may coexist with any uniform constant temperature and the wave speed depends, in general, on the wave amplitude and on the temperature.

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[2]) In fact, motions (3.1) and (4.1) and spatially uniform temperature fields may be sustained in fluids and isotropic solids, whose response is not elastic, without application of body force or radiative heat supply. A detailed treatment is given in a forthcoming paper [3].

References

[1] C. A. Truesdell and W. Noll, Handbuch der Physik III/3, Springer, Berlin (1965).
[2] M. M. Carroll, Acta Mechanica *3*, 167–181 (1967).
[3] M. M. Carroll, Arch. Rat'l Mech. Anal., to appear.

Abstract

A finite oscillatory shearing motion is shown to be possible, in the absence of body force, in every homogeneous isotropic compressible or incompressible elastic solid. The spatial geometry is the same for all materials and the nature of the time-dependence, for a particular material, is determined by the generalized shear modulus. A motion of this type and a spatially uniform, time-dependent temperature can be supported in thermoelastic solids, without application of body force or volume supply of heat.

Zusammenfassung

Es wird gezeigt, dass in Abwesenheit von Raumkräften in jedem homogenen isotropen—kompressiblen oder inkompressiblen—elastischen Festkörper eine endliche Schiebschwingung möglich ist. Die Raumgeometrie ist für alle Materialien dieselbe, und die Art der Zeitabhängigkeit wird für jedes Material durch einen verallgemeinerten Schubmodul bestimmt. Eine Bewegung dieser Art und eine räumlich gleichförmige, zeitabhängige Temperatur lassen sich in thermoelastischen Festkörpern ohne Raumkraft oder räumliche Wärmequellen aufrecht erhalten.

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