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ON THE STEADY MOTION OF VISCOUS LIQUID IN A CORNER

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Received 9 September 1948

1. In a steady two-dimensional motion of viscous liquid in the sharp corner formed by the rigid straight boundaries $\theta = 0, \alpha$, where r, θ are plane polar coordinates, it is found that, near enough to the corner, the most important term in the stream-function is of the form $r^m f(\theta)$. The index m is evaluated in §§ 2–4 for values of α between 360 and 90°, and is found to be complex if α is less than about 146°; the limiting form of the stream-function when α is small is considered in § 5.

The problem has been examined very briefly by Rayleigh (1), who assumed that m must be a multiple of π/α and concluded that a stream-function of the form considered here is possible only if α is 360 or 180°. Rayleigh's argument is considered in § 6. The result given here is, however, in agreement with that given in some papers written by the late A. C. Dixon (2), who determined Green's functions for clamped elastic plates in the forms of an infinite sector and of an infinite strip, and expressed these functions as infinite series of terms like those in §§ 3, 5.

2. In a slow two-dimensional steady motion of incompressible viscous liquid the stream-function ψ must satisfy the biharmonic equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 \psi = 0, \tag{1}$$

and, if the space in the x, y-plane is bounded by the lines $\theta = 0, \alpha$, where $x = r\cos\theta$, $y = r\sin\theta$, the boundary conditions

$$\psi = \partial\psi/\partial\theta = 0 \quad (\theta = 0, \alpha). \tag{2}$$

There are simple expressions for ψ of the form $r^m f(\theta)$ in the special cases $\alpha = \pi,\ 2\pi$, with 2 as a possible value of the index m if $\alpha = \pi$ and $\frac{3}{2}$ as a possible value if $\alpha = 2\pi$. It is accordingly natural to try a function of the same form $r^m f(\theta)$ in the general case, and to suppose that, if α lies between π and 2π, m will lie between 2 and $\frac{3}{2}$, or, if it should be complex, will have a real part between these limits; if so, we shall have the result that would have been expected on physical grounds, that the larger the angle of the corner the more rapid will be the increase in the velocity of the liquid with increasing distance from the corner.

We therefore write

$$\psi = [A\cos(n+1)\theta + B\sin(n+1)\theta + C\cos(n-1)\theta + D\sin(n-1)\theta]\, r^{n+1}, \tag{3}$$

where A, B, C, D are constants. This function is biharmonic whatever the value of n, but with some particular values of n, for instance 0 or 1, a different form of biharmonic function must be used; here it is not in fact necessary to consider these particular values. The boundary conditions (2) require that four expressions linear in the constants A, B, C, D should vanish, and if these conditions are to be satisfied by a non-zero set of constants the four-row determinant of the coefficients must be zero. As a first form of the condition for a non-zero stream-function of the type (3) we accordingly have

$$2(n^2-1)\,[1-\cos(n+1)\,\alpha\cos(n-1)\,\alpha]-2(n^2+1)\sin(n+1)\,\alpha\sin(n-1)\,\alpha = 0.$$

This condition can be written in the form

$$1-\cos 2n\alpha = n^2(1-\cos 2\alpha),$$

or

$$\sin n\alpha = \pm n\sin\alpha. \tag{4}$$

This equation determines the values of n corresponding to a given value of α.

3. The stream-function (3) is biharmonic and satisfies conditions (2) whether the index n is real or complex, but an alternative form is more useful in the latter case. In terms of z, z', where $z = x+iy$, $z' = x-iy$,

the biharmonic equation can be written

$$\frac{\partial^4\psi}{\partial z^2\partial z'^2} = 0,$$

and the boundary conditions can be replaced by the alternative conditions

$$\partial\psi/\partial z = \partial\psi/\partial z' = 0.$$

On the boundary $\theta = 0, z = z'$, and on the boundary $\theta = \alpha, z = z'e^{2i\alpha}$. We can now write

$$\psi = A_1 z^{n+1}+B_1 z'^{n+1}+C_1 z^n z'+D_1 zz'^n; \tag{5}$$

A_1, B_1, C_1 and D_1 are constants which will in general be complex, and, to assign definite values, $\exp\{n(\log r+i\theta)\}$ and $\exp\{n(\log r-i\theta)\}$ $(0 \leqslant \theta \leqslant \alpha)$,

are taken as the values of z^n and z'^n. Then

$$\partial\psi/\partial z = (n+1)\,A_1 z^n+nC_1 z^{n-1}z'+D_1 z'^n,$$
$$\partial\psi/\partial z' = (n+1)\,B_1 z'^n+C_1 z^n+nD_1 zz'^{n-1},$$

and these derivatives vanish at all points of the boundary $\theta = 0$, where $z = z'$, if

$$(n+1)\,A_1+nC_1+D_1 = 0, \quad (n+1)\,B_1+C_1+nD_1 = 0, \tag{6}$$

and at all points of the other boundary, where $z = z'e^{2i\alpha}$, if

$$(n+1)\,A_1 e^{2ni\alpha}+nC_1 e^{2(n-1)i\alpha}+D_1 = 0, \quad (n+1)\,B_1+C_1 e^{2ni\alpha}+nD_1 e^{2i\alpha} = 0. \tag{7}$$

The first form of the condition that equations (6) and (7) can be satisfied by non-zero constants A_1, B_1, C_1 and D_1 is

$$n^2\{e^{(n+1)i\alpha}-e^{(n-1)i\alpha}\}^2 = \{e^{2ni\alpha}-1\}^2,$$

and this reduces to equation (4). The form (5) for ψ will in general be complex, but all the conditions will be satisfied by both its real and its imaginary parts.

4. Equation (4) is satisfied, whether the ambiguous sign is positive or negative, by $n = 1$ if $\alpha = \pi$ and by $n = \frac{1}{2}$ if $\alpha = 2\pi$. It is first shown that for any α between π and 2π there is a real value of n between $\frac{1}{2}$ and 1. The equation can clearly be written in the form

$$\frac{\sin n\alpha}{n\alpha} = \pm\frac{\sin\alpha}{\alpha},$$

and hence a value of n other than 1 is found whenever the function $x^{-1}\sin x$ takes equal values, and whenever it takes equal and opposite values, for two different values of the

variable. Real values of n can accordingly be found from Fig. 1 which shows the graphs of $\pm x^{-1}\sin x$. It is, of course, convenient to measure angles in degrees in the numerical work; suppose then that $\alpha = 340°$ and let A be the corresponding point on the graph of $x^{-1}\sin x$. The straight line $CBAB'C'$ drawn in Fig. 1 shows that $x^{-1}\sin x$ has the same value when x is just greater than 180° as it has when $x = 340°$; hence there is a real

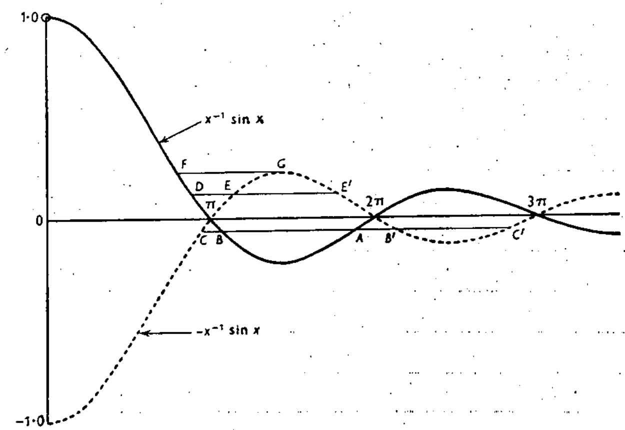

Fig. 1

value of n just greater than $\frac{1}{2}$ when $\alpha = 340°$. But there is a smaller value of n, which is also in fact greater than $\frac{1}{2}$, when $\alpha = 340°$; for Fig. 1 shows that $x^{-1}\sin x$ takes equal and opposite values when $x = 340°$ and when x has the value just less than 180° corresponding to the point C. The same line $CBAB'C'$ shows that if α has the value just greater than 180° corresponding to B, there is a possible value of n just less than 1, for $x^{-1}\sin x$ takes equal and opposite values for the values of x corresponding to B and C. If other straight lines are drawn it is easy to see that as α decreases from 340 to 180° there is a value of n increasing from the value just greater than $\frac{1}{2}$ when $\alpha = 340°$ to the value 1 when $\alpha = 180°$. Some of these values have been found numerically and are shown in the following table:

$\alpha°$	340	320	300	280	260	240	220	200
$n\alpha°$	170·1	161·1	153·7	148·5	146·3	147·8	153·4	163·7
n	0·500	0·503	0·512	0·530	0·563	0·616	0·697	0·818

If lines such as DEE' are drawn it is clear that this series of real values of n continues when α is less than 180°, the values being now greater than 1. Some of these values can be deduced at once from those given in the table; thus 200 and 163°·7 can be taken as the values of $n\alpha$ and α instead of as the values of α and $n\alpha$, so that the reciprocal of 0·818 is the value of n when α is 163°·7. The line FG gives the largest real value of n in the series; the values of α corresponding to the points F and G are 146°·3 and 257°·5 and the corresponding maximum value of n is 1·76.

For values of α less than 146°·3 the values of n satisfying equation (4) must be complex, and if

$$n = p + iq,$$

p, q must be found from the pair of equations

$$\sin p\alpha \cosh q\alpha = \pm p \sin\alpha, \quad \cos p\alpha \sinh q\alpha = \pm q \sin\alpha, \tag{8}$$

where either the two positive, or the two negative, signs are to be taken. Clearly it can be supposed that $q > 0$. The magnitude of the velocity is determined by the real part of n, and the real values should accordingly be continued by a series of values of p; if α is 140° the value of p should be about 1·8, and the negative signs in the two equations must be taken to continue the series. The following table shows some calculated values of n and p for angles α between 180 and 90°:

$\alpha°$	180	170	160	150	146·3	—
$n\alpha°$	180	191·3	206·1	230·1	257·5	—
n	1·00	1·13	1·29	1·53	1·76	—
$\alpha°$	140	130	120	110	100	90
$p\alpha°$	255·8	253·4	251·3	249·5	247·9	246·6
p	1·83	1·95	2·09	2·27	2·48	2·74

Fig. 2 shows the whole series of values of n and p for angles α between 90 and 360°.

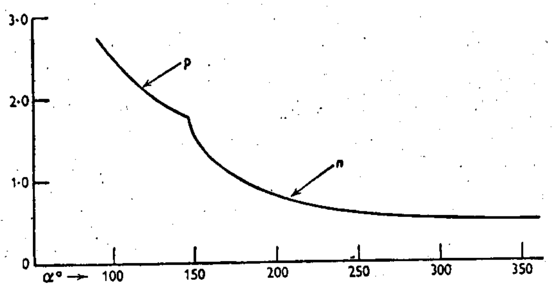

Fig. 2

5. The index n can be evaluated for angles α less than $\frac{1}{2}\pi$, but it has not appeared necessary to do more than consider the limiting case as α tends to zero, when the motion can be supposed to take place between the parallel boundaries $y = 0, a$. It is convenient to alter the origin of coordinates, and to assume that, before the limit is taken, one boundary is the line $y = 0$ and the other the line $y = a + x \tan\alpha$ passing through the point $(0, a)$; a is kept fixed as $\alpha \to 0$. Since only small angles are concerned the complex form of § 3 is appropriate. From the expression (5) for the stream-function we can derive, by using equations (6) and allowing for the change of origin, the form

$$\psi = C_2\left[\left(1+\frac{z}{a}\tan\alpha\right)^n\left\{(n+1)\left(1+\frac{z'}{a}\tan\alpha\right)-n\left(1+\frac{z}{a}\tan\alpha\right)\right\}-\left(1+\frac{z'}{a}\tan\alpha\right)^{n+1}\right]$$
$$+D_2\left[\left(1+\frac{z'}{a}\tan\alpha\right)^n\left\{(n+1)\left(1+\frac{z}{a}\tan\alpha\right)-n\left(1+\frac{z'}{a}\tan\alpha\right)\right\}-\left(1+\frac{z}{a}\tan\alpha\right)^{n+1}\right].$$

If α is small, equation (4) can be replaced by $\sin n\alpha = \pm n\alpha$, and the corresponding pair of equations (8) for a complex value of n by

$$\sin p\alpha \cosh q\alpha = \pm p\alpha, \quad \cos p\alpha \sinh q\alpha = \pm q\alpha;$$

these equations show that as α tends to zero n, p and q tend to infinity in such a way that the products $n\alpha$, $p\alpha$ and $q\alpha$ tend to finite limits which we can write Na, Pa and Qa. In the limiting case of parallel boundaries we accordingly have the equation

$$\sin Na = \pm Na,$$

and the corresponding pair of real equations

$$\sin Pa \cosh Qa = \pm Pa, \quad \cos Pa \sinh Qa = \pm Qa.$$

Since
$$\left(1+\frac{z}{a}\tan\alpha\right)^n = \exp\left[n\log\left(1+\frac{z}{a}\tan\alpha\right)\right],$$

it is clear that the limiting form, as $\alpha \to 0$, $n \to \infty$ so that $n\alpha \to Na$, is e^{Nz}, and that

$$(n+1)\left(1+\frac{z'}{a}\tan\alpha\right)-n\left(1+\frac{a}{z}\tan\alpha\right) \to 1-N(z-z');$$

evaluating the other limits we finally have

$$\psi = C_2[\{1-N(z-z')\}e^{Nz}-e^{Nz'}]+D_2[\{1+N(z-z')\}e^{Nz'}-e^{Nz}],$$

whence
$$\frac{1}{N}\frac{\partial\psi}{\partial z} = -NC_2(z-z')e^{Nz}-D_2(e^{Nz}-e^{Nz'}),$$
$$\frac{1}{N}\frac{\partial\psi}{\partial z'} = C_2(e^{Nz}-e^{Nz'})+ND_2(z-z')e^{Nz'}.$$

The constants have already been chosen so that $\partial\psi/\partial z = \partial\psi/\partial z' = 0$ when $z = z'$, and the condition that these differential coefficients should vanish, with non-zero constants C_2, D_2, on the other boundary where $z - z' = 2ia$ is easily seen to lead to the equation $\sin Na = \pm Na$, with the corresponding relation $C_2 e^{Nia} \pm D_2 = 0$ between the constants.

6. The problem of this paper of the steady motion of viscous fluid in a corner has been considered, very briefly, by Rayleigh (1). From a stream-function of the form $r^m f(\theta)$ he derived an expression equivalent to that given by equation (3), but then assumed that m must be a multiple of π/α. Rayleigh gave no reason for making this assumption, and it led him to conclude that a stream-function of the form considered here is possible only if α is a multiple of π; in practice this restricts α to the values π and 2π.

Rayleigh also considered the bending of a thin elastic plate clamped at the edges $\theta = 0, \alpha$. The transverse displacement w of the plate satisfies the same differential equation and the same boundary conditions as ψ, if the plate is supposed free from external transverse force, and Rayleigh argued that all partial differential coefficients of w with regard to x and y must vanish at the corner of the plate; this would confirm his conclusion that there cannot in general be a stream-function, or a transverse displacement, of the form $r^m f(\theta)$. Oblique Cartesian coordinates x, y were used, and $x = 0$, $y = 0$ were taken as the edges of the plate; w must then satisfy the equation

$$(\partial^2/\partial x^2+\partial^2/\partial y^2-2\cos\alpha\,\partial^2/\partial x\partial y)^2 w = 0, \tag{9}$$

and the conditions
$$w = \partial w/\partial y = 0 \quad (y = 0), \tag{10}$$
$$w = \partial w/\partial x = 0 \quad (x = 0). \tag{11}$$
It was deduced, by differentiating (10) with regard to x and (11) with regard to y, that $\partial^4 w/\partial x^4$, $\partial^4 w/\partial x^3 \partial y$, $\partial^4 w/\partial y^4$ and $\partial^4 w/\partial x \partial y^3$ vanish at the origin, and then from the differential equation (9) that $\partial^4 w/\partial x^2 \partial y^2$ also vanishes; derivatives of higher orders were proved to vanish by differentiating equation (9). From this Rayleigh concluded that the displacement at a distance r from the corner diminishes more rapidly than any power of r. The possibility that w may vary as a fractional power of r appears to be overlooked, for in such a case all differential coefficients beyond a certain order will be infinite at the origin, and the argument will not apply. It is, however, proved that w cannot be expressed as a polynomial in x and y, and this is also proved by W. Ritz (3) in a paper to which Rayleigh refers.

In a series of papers the late A. C. Dixon (2) determined Green's functions for some clamped elastic plates bounded by two or more straight edges or by two or more circular arcs; in particular, he determined these functions for an infinite sector and for an infinite strip, and they are expressed as infinite series of terms like those given in §§ 3, 5.

7. Equation (1) for the stream-function in a slow motion is obtained from the exact equation for a finite steady motion
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 \psi = \frac{1}{\nu}\left(\frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi, \tag{12}$$
where ν is the kinematic coefficient of viscosity, by neglecting the second-order inertia terms. But near enough to the corner these terms can be neglected in any steady motion since the velocity tends to zero as the corner is approached. This can also be seen by supposing that equation (12) is to be solved by successive approximation; if, for instance, $\alpha = 2\pi$ the first approximation to ψ is of order $r^{\frac{3}{2}}$ and the right-hand side of equation (12) calculated from this approximation will be of order r^{-1}, so that a term of order r^3 must be added to give the second approximation. If α is less than 2π the first approximation is in effect proportional to a power of r greater than $\frac{3}{2}$ and the same argument will apply.

It has been shown that a simple power of r is found in the first approximation only if α is π or 2π; this appears to imply that if there is a sharp corner on the boundary a simple solution of the biharmonic equation in finite terms can be expected only if the angle of the corner is 2π and not, for instance, if the corner is a right angle (4), apart from a few particular cases in which the biharmonic function may be a polynomial.

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NOTE ON THE GEOMETRICAL OPTICS OF DIFFRACTED WAVE FRONTS

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Received 17 November 1948

It is well known that a solution of the wave equation
$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$
for which $u = \partial u/\partial t = 0$ initially outside a surface S_0, vanishes at time t in the exterior of a surface S_t parallel to, and at normal distance ct from S_0, so that the wave fronts of disturbances represented by the solutions of the wave equation obey the laws of geometrical optics. Analogous results hold for the solutions of any linear hyperbolic second-order partial differential equation with boundary value conditions of the 'Cauchy' type. But the wave fronts of solutions of problems in which some of the boundary conditions are of the type representing reflexion do not seem to have been treated, and in particular the case of diffraction, when there is a 'shadow', does not seem to have been considered at all.

The present paper deals with this extension of the known properties of wave fronts to reflexion and diffraction. As the method employed is the same in all cases, it will be sufficient to treat the simple but typical case of the two-dimensional wave equation
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}, \tag{1}$$
where for simplicity the velocity of wave propagation has been taken as unity. For the purpose of this paper a solution of (1) will be required to have continuous partial derivatives of the second order. This restriction is not serious, as a physical problem can usually be reduced to this case (e.g. by repeated integration with respect to the time).

It is convenient to think of x, y and t as the coordinates of a point in a 'space-time' $\mathscr{S}$. With any point $P(x_0, y_0, t_0)$ of $\mathscr{S}$ is associated the *characteristic cone with vertex P*,
$$(x-x_0)^2+(y-y_0)^2 = (t-t_0)^2.$$
It consists of two half-cones,
$$t = t_0 - \{(x-x_0)^2+(y-y_0)^2\}^{\frac{1}{2}}, \tag{2a}$$
$$t = t_0 + \{(x-x_0)^2+(y-y_0)^2\}^{\frac{1}{2}}, \tag{2b}$$
which will be denoted by Γ_0 and Γ_0' respectively. The interior of Γ_0 is the *dependence domain* of P, the interior of Γ_0' its *influence domain*; it is obvious that a necessary and sufficient condition for a point Q to be in the dependence domain of P is that P should be in the influence domain of Q, and vice versa. This nomenclature is explained by the following important lemma, which is an easy generalization of a well-known result (1).