Finally, similar considerations can also be applied to quantities which characterize the flow but are not functions of the coordinates. Such a quantity is, for instance, the drag force F acting on the body. We can say that the dimensionless ratio of F to some quantity formed from v, u, l, ρ and having the dimensions of force must be a function of the Reynolds number alone. Such a combination of v, u, l, ρ can be $\rho u^2 l^2$, for example. Then

$$F = \rho u^2 l^2 f(\mathrm{R}). \tag{19.4}$$

If the force of gravity has an important effect on the flow, then the latter is determined not by three but by four parameters, l, u, v and the acceleration g due to gravity. From these parameters we can construct not one but two independent dimensionless quantities. These can be, for instance, the Reynolds number and the *Froude number*, which is

$$\mathrm{F} = u^2/lg. \tag{19.5}$$

In formulae (19.2)–(19.4) the function f will now depend on not one but two parameters (R and F), and two flows will be similar only if both these numbers have the same values.

Finally, we may say a little regarding non-steady flows. A non-steady flow of a given type is characterized not only by the quantities v, u, l but also by some time interval τ characteristic of the flow, which determines the rate of change of the flow. For instance, in oscillations, according to a given law, of a solid body, of a given shape, immersed in a fluid, τ may be the period of oscillation. From the four quantities v, u, l, τ we can again construct two independent dimensionless quantities, which may be the Reynolds number and the number

$$\mathrm{S} = u\tau/l, \tag{19.6}$$

sometimes called the *Strouhal number*. Similar motion takes place in these cases only if both these numbers have the same values.

If the oscillations of the fluid occur spontaneously (and not under the action of a given external exciting force), then for motion of a given type S will be a definite function of R:

$$\mathrm{S} = f(\mathrm{R}).$$

§20. **Flow with small Reynolds numbers**

The Navier–Stokes equation is considerably simplified in the case of flow with small Reynolds numbers. For steady flow of an incompressible fluid, this equation is

$$(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -(1/\rho)\mathbf{grad}\,p + (\eta/\rho)\triangle\,\mathbf{v}.$$

The term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ is of the order of magnitude of u^2/l, u and l having the same meaning as in §19. The quantity $(\eta/\rho)\triangle\,\mathbf{v}$ is of the order of magnitude of $\eta u/\rho l^2$. The ratio of the two is just the Reynolds number. Hence the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ may be neglected if the Reynolds number is small, and the equation of motion reduces to a linear equation

$$\eta\triangle\,\mathbf{v} - \mathbf{grad}\,p = 0. \tag{20.1}$$

Together with the equation of continuity

$$\mathrm{div}\,\mathbf{v} = 0 \tag{20.2}$$

it completely determines the motion. It is useful to note also the equation

$$\triangle\,\mathbf{curl}\,\mathbf{v} = 0, \tag{20.3}$$

which is obtained by taking the curl of equation (20.1).

As an example, let us consider rectilinear and uniform motion of a sphere in a viscous fluid (G. G. Stokes 1851). The problem of the motion of a sphere, it is clear, is exactly equivalent to that of flow past a fixed sphere, the fluid having a given velocity $\mathbf{u}$ at infinity. The velocity distribution in the first problem is obtained from that in the second problem by simply subtracting the velocity $\mathbf{u}$; the fluid is then at rest at infinity, while the sphere moves with velocity $-\mathbf{u}$. If we regard the flow as steady, we must, of course, speak of the flow past a fixed sphere, since, when the sphere moves, the velocity of the fluid at any point in space varies with time.

Since $\operatorname{div}(\mathbf{v}-\mathbf{u}) = \operatorname{div}\mathbf{v} = 0$, $\mathbf{v}-\mathbf{u}$ can be expressed as the curl of some vector $\mathbf{A}$:

$$\mathbf{v}-\mathbf{u} = \mathbf{curl}\,\mathbf{A},$$

with $\mathbf{curl}\,\mathbf{A}$ equal to zero at infinity. The vector $\mathbf{A}$ must be axial, in order for its curl to be polar, like the velocity. In flow past a sphere, a completely symmetrical body, there is no preferred direction other than that of $\mathbf{u}$. This parameter $\mathbf{u}$ must appear linearly in $\mathbf{A}$, because the equation of motion and its boundary conditions are linear. The general form of a vector function $\mathbf{A}(\mathbf{r})$ satisfying all these requirements is $\mathbf{A} = f'(r)\mathbf{n}\times\mathbf{u}$, where $\mathbf{n}$ is a unit vector parallel to the position vector $\mathbf{r}$ (the origin being taken at the centre of the sphere), and $f'(r)$ is a scalar function of r. The product $f'(r)\mathbf{n}$ can be represented as the gradient of another function $f(r)$. We shall thus look for the velocity in the form

$$\mathbf{v} = \mathbf{u} + \mathbf{curl}\,(\mathbf{grad}\,f\times\mathbf{u}) = \mathbf{u} + \mathbf{curl}\,\mathbf{curl}\,(f\mathbf{u}); \tag{20.4}$$

the last expression is obtained by noting that $\mathbf{u}$ is constant.

To determine the function f, we use equation (20.3). Since

$$\mathbf{curl}\,\mathbf{v} = \mathbf{curl}\,\mathbf{curl}\,\mathbf{curl}(f\mathbf{u}) = (\mathbf{grad}\,\operatorname{div} - \triangle)\,\mathbf{curl}(f\mathbf{u})$$
$$= -\triangle\,\mathbf{curl}(f\mathbf{u}),$$

(20.3) takes the form $\triangle^2\,\mathbf{curl}(f\mathbf{u}) = \triangle^2(\mathbf{grad}\,f\times\mathbf{u}) = (\triangle^2\mathbf{grad}\,f)\times\mathbf{u} = 0$. It follows from this that

$$\triangle^2\mathbf{grad}\,f = 0. \tag{20.5}$$

A first integration gives

$$\triangle^2 f = \text{constant}.$$

It is easy to see that the constant must be zero, since the velocity difference $\mathbf{v}-\mathbf{u}$ must vanish at infinity, and so must its derivatives. The expression $\triangle^2 f$ contains fourth derivatives of f, whilst the velocity is given in terms of the second derivatives of f. Thus we have

$$\triangle^2 f \equiv \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right)\triangle f = 0.$$

Hence

$$\triangle f = 2a/r + c.$$

The constant c must be zero if the velocity $v-u$ is to vanish at infinity. From $\triangle f = 2a/r$ we obtain

$$f = ar + b/r. \tag{20.6}$$

The additive constant is omitted, since it is immaterial (the velocity being given by derivatives of f).

Substituting in (20.4), we have after a simple calculation

$$\mathbf{v} = \mathbf{u} - a\frac{\mathbf{u}+\mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r} + b\frac{3\mathbf{n}(\mathbf{u}\cdot\mathbf{n})-\mathbf{u}}{r^3}. \tag{20.7}$$

The constants a and b have to be determined from the boundary conditions: at the surface of the sphere ($r = R$), $\mathbf{v} = 0$, i.e.

$$-\mathbf{u}\left(\frac{a}{R}+\frac{b}{R^3}-1\right)+\mathbf{n}(\mathbf{u}\cdot\mathbf{n})\left(-\frac{a}{R}+\frac{3b}{R^3}\right)=0.$$

Since this equation must hold for all $\mathbf{n}$, the coefficients of $\mathbf{u}$ and $\mathbf{n}(\mathbf{u}\cdot\mathbf{n})$ must each vanish. Hence $a = \frac{3}{4}R$, $b = \frac{1}{4}R^3$. Thus we have finally

$$f = \tfrac{3}{4}Rr + \tfrac{1}{4}R^3/r, \tag{20.8}$$

$$\mathbf{v} = -\tfrac{3}{4}R\frac{\mathbf{u}+\mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r} - \tfrac{1}{4}R^3\frac{\mathbf{u}-3\mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r^3}+\mathbf{u}, \tag{20.9}$$

or, in spherical polar components with the axis parallel to $\mathbf{u}$,

$$\left.\begin{aligned} v_r &= u\cos\theta\left[1-\frac{3R}{2r}+\frac{R^3}{2r^3}\right],\\ v_\theta &= -u\sin\theta\left[1-\frac{3R}{4r}-\frac{R^3}{4r^3}\right] \end{aligned}\right\} \tag{20.10}$$

This gives the velocity distribution about the moving sphere. To determine the pressure, we substitute (20.4) in (20.1):

$$\begin{aligned}\mathbf{grad}\, p = \eta\triangle\mathbf{v} &= \eta\triangle\,\mathbf{curl}\;\mathbf{curl}\;(f\mathbf{u})\\ &= \eta\triangle\,(\mathbf{grad}\;\mathrm{div}\;(f\mathbf{u}) - \mathbf{u}\triangle f).\end{aligned}$$

But $\triangle^2 f = 0$, and so

$$\mathbf{grad}\, p = \mathbf{grad}[\eta\triangle\,\mathrm{div}(f\mathbf{u})] = \mathbf{grad}(\eta\mathbf{u}\cdot\mathbf{grad}\,\triangle f).$$

Hence

$$p = \eta\mathbf{u}\cdot\mathbf{grad}\,\triangle f + p_0, \tag{20.11}$$

where p_0 is the fluid pressure at infinity. Substitution for f leads to the final expression

$$p = p_0 - \tfrac{3}{2}\eta\frac{\mathbf{u}\cdot\mathbf{n}}{r^2}R. \tag{20.12}$$

Using the above formulae, we can calculate the force $\mathbf{F}$ exerted on the sphere by the moving fluid (or, what is the same thing, the drag on the sphere as it moves through the fluid). To do so, we take spherical polar coordinates with the axis parallel to $\mathbf{u}$; by symmetry, all quantities are functions only of r and of the polar angle θ. The force $\mathbf{F}$ is evidently parallel to the velocity $\mathbf{u}$. The magnitude of this force can be determined from (15.14). Taking from this formula the components, normal and tangential to the surface, of the force on an element of the surface of the sphere, and projecting these components on the direction of $\mathbf{u}$, we find

$$F = \oint(-p\cos\theta + \sigma'_{rr}\cos\theta - \sigma'_{r\theta}\sin\theta)\mathrm{d}f, \tag{20.13}$$

where the integration is taken over the whole surface of the sphere.

Substituting the expressions (20.10) in the formulae

$$\sigma'_{rr} = 2\eta\frac{\partial v_r}{\partial r}, \qquad \sigma'_{r\theta} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

(see (15.20)), we find that at the surface of the sphere

$$\sigma'_{rr} = 0, \qquad \sigma'_{r\theta} = -(3\eta/2R)u\sin\theta,$$

while the pressure (20.12) is $p = p_0 - (3\eta/2R)u\cos\theta$. Hence the integral (20.13) reduces to $F = (3\eta u/2R)\oint \mathrm{d}f$. In this way we finally arrive at *Stokes' formula* for the drag on a sphere moving slowly in a fluid:†

$$F = 6\pi\eta Ru. \tag{20.14}$$

The drag is proportional to the velocity and linear size of the body. This could have been foreseen from dimensional arguments: the fluid density ρ does not appear in the approximate equations (20.1), (20.2), and so the force F which they give must be expressed only in terms of η, u and R; from these, only one combination with the dimensions of force can be formed, namely the product ηRu.

A similar dependence occurs for slowly moving bodies with other shapes. The direction of the drag on a body of arbitrary shape is not the same as that of the velocity; the general form of the dependence of $\mathbf{F}$ on $\mathbf{u}$ can be written

$$F_i = \eta a_{ik} u_k, \tag{20.15}$$

where a_{ik} is a tensor of rank two, independent of the velocity. It is important to note that this tensor is symmetrical, a result which holds in the linear approximation with respect to the velocity, and is a particular case of a general law valid for slow motion accompanied by dissipative processes (see *SP*1, §121).

REFINEMENT OF STOKES' FORMULA

The above solution of the problem of flow past a sphere is not valid at large distances, even if the Reynolds number is small. To see this, let us estimate the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ neglected in (20.1). At large distances, $\mathbf{v} \cong \mathbf{u}$; the velocity derivatives there are of the order of uR/r^2, as is seen from (20.9). Hence $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} \sim u^2R/r^2$. The terms retained in (20.1) are of the order of $\eta Ru/\rho r^3$, as can be seen from the same expression (20.9) for the velocity or (20.12) for the pressure. The condition $\eta Ru/\rho r^3 \gg u^2R/r^2$ is satisfied only for distances such that

$$r \ll \nu/u. \tag{20.16}$$

At greater distances, the terms neglected are not negligible, and the velocity distribution so found is incorrect.

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† With a view to later applications, it may be mentioned that calculations with (20.7) and the constants a and b undetermined give

$$F = 8\pi\eta au. \tag{20.14a}$$

The drag can also be calculated for a slowly moving ellipsoid with any shape. The relevant formulae are given by H. Lamb, *Hydrodynamics*, 6th ed., §339, Cambridge 1932. Here we shall give the limiting expressions for a plane circular disk with radius R moving perpendicular to its plane:

$$F = 16\eta Ru,$$

and for a similar disk moving in its plane:

$$F = 32\eta Ru/3.$$