A dichotomy for countable unions of smooth Borel equivalence relations

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Caltech logic seminar

April 26, 2021

Borel reducibility

Definition

Let E and F be two equivalence relations on sets X and Y, respectively.

- A mapping $f\colon X \to Y$ is a reduction from E to F if it induces an injection $X/E \to Y/F$. Equivalentely, for all $x, x' \in X$, $x \mathrel{E} x' \Leftrightarrow f(x) \mathrel{F} f(x')$.
- An embedding is a one-to-one reduction.

Definition

Let E and F be two equivalence relations on Polish spaces X and Y, respectively.

- We say that E Borel reduces to F, denoted by $E \leqslant_B F$, if there is a Borel reduction from E to F.
- We write $E \equiv_B F$ to say that $E \leqslant_B F$ and $F \leqslant_B E$, and $E <_B F$ to say that $E \leqslant_B F$ and $F \nleqslant_B E$.
- We say that E continuously embeds into F, denoted by $E \sqsubseteq_c F$, if there is a continuous embedding from E to F.

Some classical Borel equivalence relations

Given a Polish space X, denote equality on X by Δ_X.

Definition

A Borel equivalence relation on a Polish space is said to be:

- countable if its classes are countable;
- smooth if it Borel reduces to equality on a Polish space;
- essentially countable if it Borel reduces to a countable Borel equivalence relation on a Polish space.

An example of a countable, non-smooth Borel equivalence relation is the relation $\mathbb{E}_0$ on $2^{\mathbb{N}}$ given by $x \, \mathbb{E}_0 \, y$ iff $x(n) = y(n)$ eventually.

We have the following initial segment of the hierarchy of Borel equivalence relations:

$$\Delta_1 <_B \Delta_2 <_B \ldots <_B \Delta_{\mathbb{N}} <_B \Delta_{\mathbb{R}} <_B \mathbb{E}_0,$$

which is exhaustive in the sense that every Borel equivalence relation is either bireducible with one of the elements of this initial segment, or is strictly greater than $\mathbb{E}_0$.

Hypersmooth Borel equivalence relations

Definition

Say that a Borel equivalence relation on a Polish space is hypersmooth if it can be written as a countable increasing union of smooth Borel equivalence relations.

The relation Δ_X for every X, and $\mathbb{E}_0$, are hypersmooth. Another example is the equivalence relation $\mathbb{E}_1$ on $(2^{\mathbb{N}})^{\mathbb{N}}$ defined by $x \mathbb{E}_1 y$ iff $x(n) = y(n)$ eventually.

It is easy to see that a Borel equivalence E is hypersmooth iff $E \leqslant_B \mathbb{E}_1$.

Proposition (Folklore)

The relation $\mathbb{E}_1$ is not essentially countable.

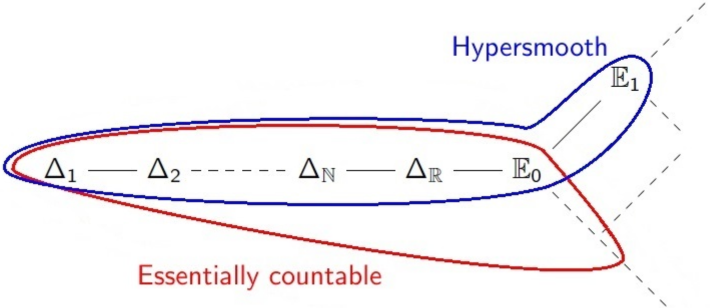
Kechris–Louveau's dichotomy

Theorem (Kechris–Louveau)

Let E be a Borel hypersmooth equivalence relation on a Polish space. Then exactly one of the two following conditions holds:

- $E \leqslant_B \mathbb{E}_0$;
- $\mathbb{E}_1 \sqsubseteq_c E$.

In particular, $\mathbb{E}_1$ is an immediate successor of $\mathbb{E}_0$ under $\leqslant_B$.



σ-smoothness

Definition

A Borel equivalence relation E on a Polish space is said to be σ-smooth if it is a countable union of smooth Borel subequivalence relations.

Hypersmooth Borel equivalence relations are obviously σ-smooth.

Lemma

Essentially countable Borel equivalence relations are σ-smooth.

Proof.

It is enough to prove it for countable Borel equivalence relations. By the proof of Feldman-Moore's theorem, they can be expressed as countable unions of graphs of Borel involutions. Those graphs are finite, hence smooth equivalence relations. □

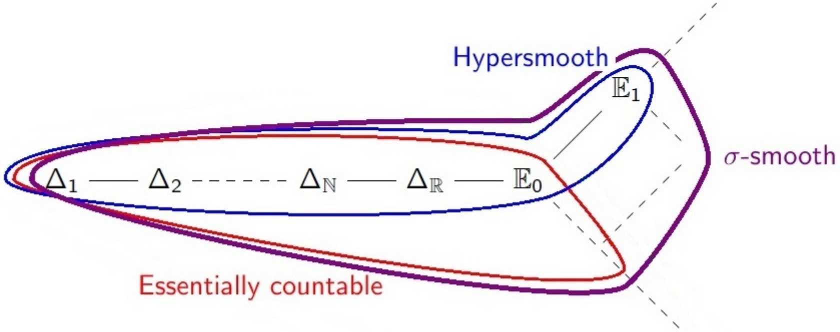
There are other examples, for instance the disjoint union of $\mathbb{E}_1$ and of a non-hypersmooth countable Borel equivalence relation.

The main theorem

Theorem

Let E be σ-smooth Borel equivalence relation on a Polish space. Then exactly one of the following conditions holds.

- *E is essentially countable.*
- $\mathbb{E}_1 \sqsubseteq_c E$.



Idealisticity

A Borel equivalence relation E on a Polish space X is said to be idealistic (resp. strongly idealistic) if there is an E-invariant assignment $x \mapsto \mathcal{I}_x$ mapping each point in X to a σ-ideal on X in such a way that:

- $\forall x \in X, \ [x]_E \notin \mathcal{I}_x$;
- For every Borel set $R \subseteq X \times X$, the set $\{x \in X \mid R_x \in \mathcal{I}_x\}$ is Borel (resp. for every Polish space Y and every Borel set $R \subseteq X \times Y \times X$, the set $\{(x, y) \in X \times Y \mid R_{x,y} \in \mathcal{I}_x\}$ is Borel).

The equivalence relation E is said to be ccc idealistic (resp. strongly ccc idealistic) if for every $x \in X$ and every uncountable family $(B_i)_{i \in I}$ of pairwise disjoint Borel subsets of X, one of the B_i's is in $\mathcal{I}_x$.

Group actions

Given a Borel action $G \curvearrowright X$ of a Polish group on a Polish space, we can consider the orbit equivalence relation associated to this action, i.e. the analytic equivalence relation E_G^X on X defined by:

$$x \, E_G^X \, x' \Leftrightarrow (\exists g \in G)(g \cdot x = x').$$

Proposition (Folklore)

Borel orbit equivalence relations on Polish spaces are strongly ccc idealistic.

Theorem (Feldman–Moore)

Let E be a countable Borel equivalence relation on a Polish space X. Then there is a Borel action $\Gamma \curvearrowright X$ of a countable discrete group such that $E = E_\Gamma^X$.

Idealisticity and $\mathbb{E}_1$

Theorem (Kechris–Louveau)

$\mathbb{E}_1$ *is not Borel reducible to any ccc idealistic Borel equivalence relation.*

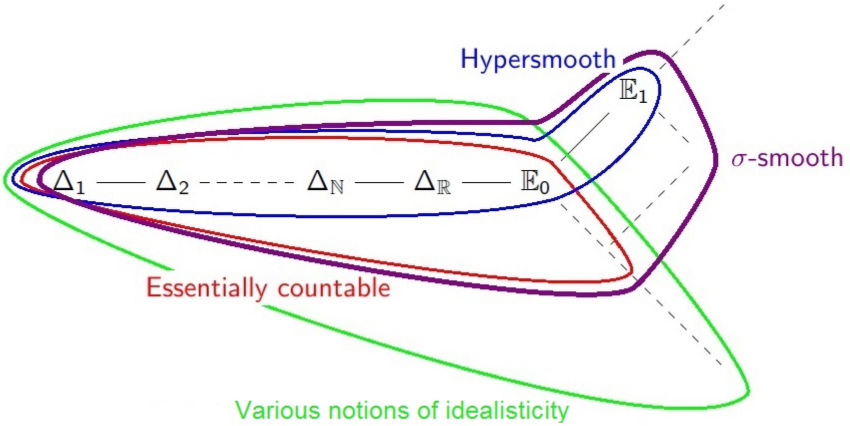
Conjecture (Kechris–Louveau)

Let E be a Borel equivalence relation on a Polish space. Then exactly one of the following two conditions holds:

- *E Borel reduces to a ccc idealistic Borel equivalence relation on a Polish space;*
- $\mathbb{E}_1 \sqsubseteq E$.

Kechris–Louveau's dichotomy solves this conjecture in the special case of hypersmooth Borel equivalence relations. Our dichotomy solves it in the special case of σ-smooth Borel equivalence relations.

A picture



Hypersmooth
$\mathbb{E}_1$
σ-smooth
Δ_1 — Δ_2 - - - - - - $\Delta_\mathbb{N}$ — $\Delta_\mathbb{R}$ — $\mathbb{E}_0$
Essentially countable
Various notions of idealisticity

Stability under countable unions

Theorem (Rephrasing of the main theorem)

Let E be a Borel equivalence relation on a Polish space. Suppose that $\mathbb{E}_1 \not\leq_B E$ (this holds, for instance, if E is ccc idealistic). If E is a countable union of essentially countable Borel subequivalence relations, then E is essentially countable.

We want to generalize this result to other classes than the class of countable Borel equivalence relations.

Definition

An equivalence relation E on a Polish space X is said to be potentially F_σ if it is Borel reducible to an F_σ equivalence relation on a Polish space.

Stability under countable unions

Definition

For every $n \in \mathbb{N}$, let E_n be an equivalence relation on a set X_n. The disjoint union of the E_n's is the equivalence relation E on $X := \bigsqcup_{n \in \mathbb{N}} X_n$ defined by $x \mathrel{E} x' \Leftrightarrow (\exists n \in \mathbb{N})(x, x' \in X_n \text{ and } x \mathrel{E_n} x')$.

Definition

Let $E \subseteq F$ be two equivalence relations on the same set X. Say that F has countable index over E if each F-class is a countable union of E-classes.

If $\mathcal{F}$ is a family of Borel equivalence relations on Polish spaces, denote by $\mathcal{F}^{\leqslant_B}$ the family of all equivalence relations on Polish spaces that are Borel reducible to an element of $\mathcal{F}$, and by $\sigma(\mathcal{F})$ the class of all equivalence relations on Polish spaces that can be expressed as countable unions of subequivalence relations belonging to $\mathcal{F}$.

Our most general result

Theorem

Let $\mathcal{F}$ be a class of strongly idealistic potentially F_σ equivalence relations on Polish spaces. Suppose that $\mathcal{F}$ is closed under countable disjoint unions and countable index Borel superequivalence relations. Let $E \in \sigma(\mathcal{F}^{\leqslant_B})$. Then at least one of the following conditions holds:

- $E \in \mathcal{F}^{\leqslant_B}$;
- $\mathbb{E}_1 \sqsubseteq_c E$.

Moreover, if elements of $\mathcal{F}$ are ccc idealistic, then these two conditions are mutually exclusive.

Our dichotomy for σ-smooth equivalence relations is the special case when $\mathcal{F}$ is the class of all countable Borel equivalence relations.

A consequence

When $\mathcal{F}$ is the class of strongly ccc idealistic potentially F_σ equivalence relations on Polish spaces, we obtain:

Corollary

Let E be an equivalence relation on a Polish space. Suppose that E can be expressed as a countable union of subequivalence relations that are Borel reducible to strongly ccc idealistic potentially F_σ equivalence relations on Polish spaces. Then exactly one of the following conditions hold:

- *E is Borel reducible to a strongly ccc idealistic potentially F_σ equivalence relation on a Polish space.*
- $\mathbb{E}_1 \sqsubseteq_c E$.

This proves Kechris–Louveau's conjecture for the class of equivalence relations that can be expressed as countable unions of subequivalence relations that are Borel reducible to strongly ccc idealistic potentially F_σ equivalence relations on Polish spaces.

F^+ and $F^\cap$

Definition

Let F be an equivalence relation on a Polish space X.

- For $A \subseteq X$, denote by $[A]_F$ the F-saturation of A, that is, the set $\{x \in X \mid (\exists x' \in A)(x \mathrel{F} x')\}$.
- The Friedman–Stanley jump of F is the equivalence relation F^+ on $X^\mathbb{N}$ defined by $x \mathrel{F^+} x'$ iff $[x(\mathbb{N})]_F = [x'(\mathbb{N})]_F$.
- The binary relation $F^\cap$ on $X^\mathbb{N}$ is defined by $x \mathrel{F^\cap} x'$ iff $[x(\mathbb{N})]_F \cap [x'(\mathbb{N})]_F$ is nonempty.

Definition

- A homomorphism from a binary relation R on a set X to a binary relation S on a set Y is a mapping $f \colon X \to Y$ such that $(f \times f)[R] \subseteq S$.
- A reduction from R to S is a mapping $f \colon X \to Y$ which is both a homomorphism from R to S and from $\sim R$ to $\sim S$.

F^+ and $F^\cap$

Proposition

Let E be an equivalence relation on a Polish space X, and F be a strongly idealistic Borel equivalence relation on a Polish space Y. Suppose that E Borel reduces to a countable-index superequivalence relation of F. Then there is a Borel homomorphism from $(E, {\sim}E)$ to $(F^+, {\sim}F^\cap)$. In particular, E Borel reduces to $F^\cap$.

Proposition

Let E and F be Borel equivalence relations on Polish spaces X and Y, respectively. The following are equivalent:

- *E Borel reduces to $(F \times \Delta_{\mathbb{N}})^\cap$;*
- *E is a countable union of subequivalence relations that are Borel reducible to $F \times \Delta_{\mathbb{N}}$.*

Another consequence

Theorem

Let E be an equivalence relation on a Polish space which is Borel reducible to a ccc idealistic Borel equivalence relation. Let $\tilde{F}$ be a strongly idealistic potentially F_σ equivalence relation on a Polish space and let $F = \tilde{F} \times \Delta_{\mathbb{N}}$. The following are equivalent:

- *E is Borel reducible to a countable index superequivalence relation of F;*
- *There is a Borel homomorphism from $(E, \sim E)$ to $(F^+, \sim F^\cap)$;*
- *E Borel reduces to $F^\cap$;*
- *E is a countable union of subequivalence relations that are Borel reducible to F.*

Moreover, if these conditions are satisfied, then $E \leqslant_B F^+$.

Thank you for your attention!