Weak Fraïssé classes and $\aleph_0$-categoricity

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The setting

We work in a fixed finite and relational language $\mathcal{L}$.

All classes of finite structures will be supposed to be closed under isomorphism, hereditary, to satisfy the joint embedding property, and to contain arbitrarily large structures.

- Hereditary: if $\mathcal{A} \in \mathfrak{F}$ and $\mathcal{B} \leqslant \mathcal{A}$ then $\mathcal{B} \in \mathfrak{F}$.
- Joint embedding property: if $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$ then there exists $\mathcal{C} \in \mathfrak{F}$ into which both $\mathcal{A}$ and $\mathcal{B}$ can embed.

If $\mathcal{X}$ is an (infinite) countable structure, denote by $\mathrm{Age}(\mathcal{X})$ the class of all finite structures that are embeddable into $\mathcal{X}$.

If $\mathfrak{F}$ is a class of finite structures, denote by $\sigma\mathfrak{F}$ the class of all countable structures $\mathcal{X}$ with $\mathrm{Age}(\mathcal{X}) \subseteq \mathfrak{F}$.

Question

When does $\sigma\mathfrak{F}$ have a generic element, and how does this element look like?

Mod($\mathcal{L}$)

Denote by Mod($\mathcal{L}$) the set of all countable structures with underlying set $\mathbb{N}$.

Mod($\mathcal{L}$) can be endowed with a compact Polish space structure, by identifying it with $\prod_{i<n} 2^{\mathbb{N}^{a_i}}$, where $\mathcal{L} = \{R_i \mid i < n\}$ and R_i has arity a_i.

$\sigma_{\mathbb{N}}\mathfrak{F} := \sigma\mathfrak{F} \cap \mathrm{Mod}(\mathcal{L})$ is a closed subset of Mod($\mathcal{L}$).

For $\mathcal{X}$ countable, let $\langle \mathcal{X} \rangle := \{\mathcal{Y} \in \mathrm{Mod}(\mathcal{L}) \mid \mathcal{Y} \cong \mathcal{X}\}$.

Observation

$\sigma_{\mathbb{N}}\,\mathrm{Age}(\mathcal{X}) = \overline{\langle \mathcal{X} \rangle}$.

Theorem (Cameron 1991)

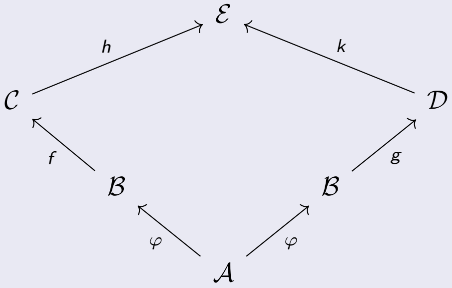
If $\mathfrak{F}$ is a Fraïssé class, then its Fraïssé limit has a comeager isomorphism class in $\sigma\mathfrak{F}$.

Weak amalgamation

We work in the category of all $\mathcal{L}$-structures with embeddings as arrows.

Definition

An arrow $\varphi\colon \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$, is $\mathfrak{F}$-amalgamable if $\forall \mathcal{C}, \mathcal{D} \in \mathfrak{F}$, $\forall f\colon \mathcal{B} \to \mathcal{C}$, $\forall g\colon \mathcal{B} \to \mathcal{D}$, $\exists \mathcal{E} \in \mathfrak{F}$, $\exists h\colon \mathcal{C} \to \mathcal{E}$, $\exists k\colon \mathcal{D} \to \mathcal{E}$ such that the following diagram commutes:



Weak amalgamation

Say that the class $\mathfrak{F}$:

- satisfies the amalgamation property (AP) if for every $\mathcal{A} \in \mathfrak{F}$, $\mathrm{Id}_{\mathcal{A}}$ is $\mathfrak{F}$-amalgamable;
- satisfies the cofinal amalgamation property (CAP) if for every $\mathcal{A} \in \mathfrak{F}$, there exists $\mathcal{B} \in \mathfrak{F}$ with $\mathcal{A} \leqslant \mathcal{B}$ such that $\mathrm{Id}_{\mathcal{B}}$ is $\mathfrak{F}$-amalgamable;
- satisfies the weak amalgamation property (WAP) if for every $\mathcal{A} \in \mathfrak{F}$, there exists an $\mathfrak{F}$-amalgamable arrow with domain $\mathcal{A}$.

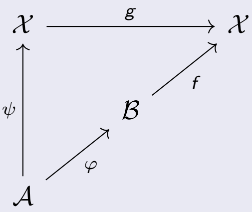
Theorem (Krawczyk–Kubiś 2021)

$\sigma_{\mathbb{N}}\mathfrak{F}$ contains a comeager isomorphism class if and only if the class $\mathfrak{F}$ has the WAP.

Weak/pre-(ultra)homogeneity

Definition

A countable structure $\mathcal{X}$ is said to be weakly ultrahomogeneous (or weakly homogeneous, or prehomogeneous) if for every $\mathcal{A} \in \mathrm{Age}(\mathcal{X})$ and every $\psi\colon \mathcal{A} \to \mathcal{X}$, there exists $\mathcal{B} \in \mathrm{Age}(\mathcal{X})$ and $\varphi\colon \mathcal{A} \to \mathcal{B}$ such that for every $f\colon \mathcal{B} \to \mathcal{X}$, there exists $g \in \mathrm{Aut}(\mathcal{X})$ such that the following diagram commutes:

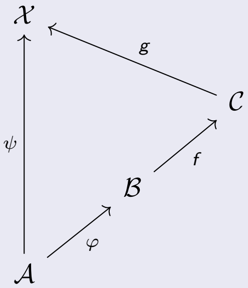


Such a φ will be called a ψ-homogeneity witness.

Weak injectivity

Definition

A countable structure $\mathcal{X}$ is said to be weakly injective if for every $\mathcal{A} \in \mathrm{Age}(\mathcal{X})$ and every $\psi\colon \mathcal{A} \to \mathcal{X}$, there exists $\mathcal{B} \in \mathrm{Age}(\mathcal{X})$ and $\varphi\colon \mathcal{A} \to \mathcal{B}$ such that for every $\mathcal{C} \in \mathrm{Age}(\mathcal{X})$ and every $f\colon \mathcal{B} \to \mathcal{C}$, there exists $g\colon \mathcal{C} \to \mathcal{X}$ such that the following diagram commutes:



Such a φ will be called a ψ-injectivity witness.

Weak ultrahomogeneity and weak injectivity

Theorem (Krawczyk–Kubiś 2021)

A countable structure $\mathcal{X}$ is wealkly injective if and only if it is weakly ultrahomogeneous. Moreover, for $\mathcal{A} \in \mathrm{Age}(\mathcal{X})$ and $\psi\colon \mathcal{A} \to \mathcal{X}$, ψ-injectivity witnesses and ψ-homogeneity witnesses coincide.

Let $\mathcal{X}$ be a weakly ultrahomogeneous structure. $\mathcal{X}$ is said to be:

- ultrahomogeneous if for every finite $\mathcal{A} \leqslant \mathcal{X}$, Id_A is a homogeneity witness for the inclusion map $\mathcal{A} \hookrightarrow \mathcal{X}$.
- cofinally ultrahomogeneous if for every finite $\mathcal{A} \leqslant \mathcal{X}$, there exists a finite $\mathcal{B}$ with $\mathcal{A} \leqslant \mathcal{B} \leqslant \mathcal{X}$ such that Id_B is a homogeneity witness for the inclusion map $\mathcal{B} \hookrightarrow \mathcal{X}$.

The Fraïssé correspondance

Theorem (Pouzet–Roux 1996, Krawczyk–Kubiś 2021)

If a countable structure $\mathcal{X}$ is weakly ultrahomogeneous, then $\mathrm{Age}(\mathcal{X})$ has the WAP. Conversely, if a class $\mathfrak{F}$ of finite structures has the WAP, then there exists a countable weakly ultrahomogeneous structure $\mathcal{X}$, unique up to isomorphism, such that $\mathrm{Age}(\mathcal{X}) = \mathfrak{F}$. This structure moreover satisfies:

(1) $\langle \mathcal{X} \rangle$ is a dense G_δ subset of $\sigma_{\mathbb{N}}\mathfrak{F}$;

(2) for every $\mathcal{A}, \mathcal{B} \in \mathrm{Age}(\mathcal{X})$, every $\psi \colon \mathcal{A} \to \mathcal{X}$ and every $\varphi \colon \mathcal{A} \to \mathcal{B}$, the arrow φ is a ψ-homogeneity witness iff it is an $\mathfrak{F}$-amalgamable arrow and ψ factors through φ.

The structure $\mathcal{X}$ is called the (weak) Fraïssé limit of $\mathfrak{F}$ and denoted by $\mathrm{Flim}(\mathfrak{F})$. It follows from (2) that $\mathrm{Flim}(\mathfrak{F})$ is ultrahomogeneous (resp. cofinally ultrahomogeneous) iff $\mathfrak{F}$ has the AP (resp. the CAP).

Genericity

Theorem (Pouzet–Roux 1996)

If $\mathcal{X}$ is a countable structure such that $\langle \mathcal{X} \rangle$ is comeager in its closure in $\mathrm{Mod}(\mathcal{L})$*, then $\mathcal{X}$ is weakly ultrahomogeneous.*

Summary

To summarize:

- We have a Fraïssé correspondance between classes with the WAP and weakly ultrahomogeneous structures. It generalizes the usual Fraïssé correspondance.
- For a countable structure $\mathcal{X}$, the following are equivalent:

  (1) $\mathcal{X}$ is weakly ultrahomogeneous;

  (2) $\mathcal{X}$ is weakly injective;

  (3) $\langle\mathcal{X}\rangle$ is a G_δ subset of $\mathrm{Mod}(\mathcal{L})$;

  (4) $\langle\mathcal{X}\rangle$ is comeager in its closure in $\mathrm{Mod}(\mathcal{L})$.

$\aleph_0$-categoricity

Definition

A countable structure $\mathcal{X}$ is said to be $\aleph_0$-categorical if it is countable and there exists a first-order theory T whose only countable model is $\mathcal{X}$, up to isomorphism.

Theorem (Engeler–Ryll-Nardzewski–Svenonius)

Let $\mathcal{X}$ be a countable structure. The following are equivalent:

(1) *$\mathcal{X}$ is $\aleph_0$-categorical;*

(2) *for every $n \in \mathbb{N}$, $\mathcal{X}$ has finitely many n-types without parameters;*

(3) *for every $n \in \mathbb{N}$, $\mathcal{X}^n /\!\!/ \mathrm{Aut}(\mathcal{X})$ is finite.*

Homogeneity properties and $\aleph_0$-categoricity

Fact

If $\mathcal{X}$ is ultrahomogeneous, then $\mathcal{X}$ is $\aleph_0$-categorical.

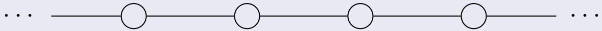
There are no other implications between a homogeneity property and $\aleph_0$-categoricity.

Example

Let $\overline{\mathbb{Q}} := \mathbb{Q} \cup \{\pm\infty\}$. Then $(\overline{\mathbb{Q}}, <)$ is $\aleph_0$-categorical, but not weakly ultrahomogeneous.

Example

Consider the following graph Z:



Then Z is not $\aleph_0$-categorical, but it is cofinally homogeneous.

$\aleph_0$-categorical weakly ultrahomogeneous structures

If $\mathcal{X}$ is weakly ultrahomogeneous and $\mathcal{A} \in \mathrm{Age}(\mathcal{X})$, then the type of an embedding $\psi\colon \mathcal{A} \to \mathcal{X}$ is entirely determined by a ψ-injectivity witness.

Theorem

Let $\mathfrak{F}$ be a class of finite structures with the WAP and let $\mathcal{X} := \mathrm{Flim}(\mathfrak{F})$. The following are equivalent:

(1) *$\mathcal{X}$ is $\aleph_0$-categorical;*

(2) *for every $\mathcal{A} \in \mathfrak{F}$, there is a finite family $\mathfrak{B}_{\mathcal{A}}$ of amalgamable arrows with domain $\mathcal{A}$ such that each $\psi\colon \mathcal{A} \to \mathcal{X}$ has an injectivity witness in $\mathfrak{B}_{\mathcal{A}}$.*

$\aleph_0$-categorical weakly ultrahomogeneous structures

Definition

Say that a class $\mathfrak{F}$ of finite structures satisfies the uniform WAP if it satisfies the WAP and condition *(2)* from last slide.

Theorem

The uniform WAP can be read on the class in a finitary way.

Age of an $\aleph_0$-categorical structure

Theorem

Let $\mathcal{X}$ be an $\aleph_0$-categorical structure. Then $\mathrm{Age}(\mathcal{X})$ *has the WAP and* $\mathrm{Flim}(\mathrm{Age}(\mathcal{X}))$ *is also $\aleph_0$-categorical.*

Definition

A countable structure $\mathcal{X}$ is said to be universal if every element of $\sigma\,\mathrm{Age}(\mathcal{X})$ can be embedded into $\mathcal{X}$.

The WAP for $\mathrm{Age}(\mathcal{X})$ relies on the two following results.

Theorem (Folklore? Seen in Cameron 1991)

$\aleph_0$-categorical structures are universal.

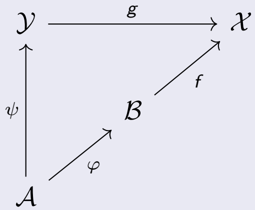
Theorem (Krawczyk–Kubiś 2021)

If $\mathcal{X}$ is universal, then $\mathrm{Age}(\mathcal{X})$ *has the WAP.*

Strong universality

Definition

A countable structure $\mathcal{X}$ is said to be strongly universal if for every countable structure $\mathcal{Y}$, every $\mathcal{A} \in \mathrm{Age}(\mathcal{Y})$, and every $\psi\colon \mathcal{A} \to \mathcal{Y}$, there exists $\mathcal{B} \in \mathrm{Age}(\mathcal{Y})$ and $\varphi\colon \mathcal{A} \to \mathcal{B}$ such that for every $f\colon \mathcal{B} \to \mathcal{X}$, there exists an embedding $g\colon \mathcal{Y} \to \mathcal{X}$ such that the following diagram commutes:



Such a φ will be called an ψ-universality witness.

Strong universality

Strong universality implies universality.

Theorem

Every $\aleph_0$-categorical structure is stronlgy universal.

Strong universality is difficult to handle and very different from weak ultrahomogeneity. For instance, even if $\mathrm{Age}(X)$ satisfies the AP, identity arrows cannot always be universality witnesses.

Question

Are there non-$\aleph_0$-categorical, strongly universal structures?

Sketch of the proof of the main theorem

If $\mathcal{X}$ is a countable structure and $\mathcal{A} \in \mathrm{Age}(\mathcal{X})$, then for $f, g \colon \mathcal{A} \to \mathcal{X}$, we write $f \leqslant g$ if there exists an embedding $h \colon \mathcal{X} \to \mathcal{X}$ such that $g = h \circ f$. This is a quasiordering, denote by $\equiv$ the associated equivalence relation. If X is $\aleph_0$-categorical, then $\equiv$ has finitely many classes. In particular, the quasiordering $\leqslant$ has maximal elements.

Lemma

Let $\psi \colon \mathcal{A} \to \mathcal{X}$ be $\leqslant$-maximal. Let φ be a ψ-universality witness. Then φ is an $\mathrm{Age}(\mathcal{X})$-amalgamable arrow.

Taking one arrow of this form for each maximal class is enough to witness that $\mathrm{Age}(\mathcal{X})$ satisfies the uniform WAP.

An open question

Question

Is there a condition (C) on countable structures, expressible in category-theoretic terms, such that for every class $\mathfrak{F}$ with the uniform WAP, structures $\mathcal{Y} \in \sigma\mathfrak{F}$ are $\aleph_0$-cateogrical iff they satisfy (C)?

Could this condition (C) be strong universality?

Link with model-companionship

Definition

A theory T is said to be model-complete if every embedding between models of T is elementary.

Definition

Let T be a theory. A model-companion of T is a theory T' such that:

(1) T' is model-complete;

(2) every model of T embeds into a model of T';

(3) every model of T' embeds into a model of T.

If a model companion exists, then it is unique.

Link with model-companionship

Fact

Let $\mathcal{X}$ be an $\aleph_0$-categorical, weakly ultrahomogeneous structure. Then $\mathrm{Th}(\mathcal{X})$ *is model-complete.*

Using the main theorem, the latter fact, and universality, we easily obtain:

Theorem (Saracino 1973)

Let T be an $\aleph_0$-categorical theory. Then T has a model companion, which is also $\aleph_0$-categorical.

Thank you for your attention!