Test TMI1 21.12.2020

1. Vyjádřete následující integrál jako součet číselné řady:

$$\int_0^1 x \log x \log(1+x)\, dx.$$

2. Najděte definiční obor a spočtěte

$$F(a) = \int_0^\infty \frac{e^{-x} - e^{-ax}}{x}\, dx.$$

3. Spočtěte Lebesgueovu míru množiny

$$\{(x,y,z) : x^2 + y^2 < 2x,\ x < z < 4\}.$$

Všechny výpočty řádně zdůvodněte, ověřte předpoklady vět, které používáte.

Test 21.12.2020

1) $\int_0^1 x \log x \log(1+x)\, dx$

$$= \int_0^1 x \log x \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}\, dx$$

$$= \int_0^1 \sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n-1}}{n} x^{n+1} \log x}_{f_n(x)}\, dx$$

$$\int_0^1 f_n(x)\, dx = \frac{(-1)^{n-1}}{n} \int_0^1 x^{n+1} \log x\, dx$$

$$= \frac{(-1)^{n-1}}{n} \int_0^1 - \frac{x^{n+1}}{n+2}\, dx \qquad \left[\begin{array}{c|c} u' = x^{n+1} & v = \log x \\ u = \frac{x^{n+2}}{n+2} & v' = \frac{1}{x} \end{array} \right]$$

$$= \frac{(-1)^n}{n(n+2)^2}$$

$$\sum_{n=1}^{\infty} \int_0^1 |f_n(x)|\, dx = \sum_{n=1}^{\infty} \frac{1}{n(n+2)^2} < \infty$$

$$\Rightarrow \int_0^1 \sum_{n=1}^{\infty} f_n(x)\, dx = \sum_{n=1}^{\infty} \int f_n(x)\, dx = \underline{\underline{\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+2)^2}}}$$

2

2) $$F(a) = \int_0^\infty \underbrace{\frac{e^{-x} - e^{-ax}}{x}}_{f(a,x)} \, dx$$

- $a < 0 \Rightarrow$ int. diverguje v ∞:

$$\lim_{x \to \infty} f(a,x) = -\infty$$

- $a = 0 \Rightarrow$ int. diverguje v ∞ (srovnáme s $g(x) = \frac{1}{x}$)

- $\underline{a > 0}$:

$$F(a) = \int_0^\infty \left[\frac{e^{-yx}}{x} \right]_{y=a}^{y=1} dx$$

$$= \int_0^\infty \int_1^a e^{-yx} \, dy \, dx$$

(Fubini: nezáp. měř. funkce) $$= \int_1^a \int_0^\infty e^{-yx} \, dx \, dy$$

$$= \int_1^a \left[\frac{e^{-yx}}{-y} \right]_{x=0}^{x=\infty} dy$$

$$= \int_1^a \frac{1}{y} \, dy = \underline{\underline{\log a}}$$

3) $T = \{(x,y,z) : x^2+y^2 < 2x,\ x < z < 4\}$ $\llcorner 3$

válcové souřadnice:

$$\varphi : \begin{cases} x = r\cos t \\ y = r\sin t \\ z = z \end{cases} \qquad J\varphi(r,t,z) = r$$

$$\varphi^{-1}(T) = \{(r,t,z) : r>0,\ r^2 < 2r\cos t,\ r\cos t < z < 4,\ t \in (-\pi,\pi)\}$$

$\cdot \cos t > 0 \implies \underline{t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

$$\left(\varphi^{-1}(T)\right)_{r,t} = (r\cos t, 4) \qquad (\text{řez}) \qquad \left(\begin{matrix} r\cos t < 4 \\ r < 2 \end{matrix}\right)$$

$$\lambda^3(T) = \iint\limits_{\left\{\begin{matrix} r < 2\cos t \\ r>0 \\ t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\end{matrix}\right\}} r \int_{r\cos t}^{4} dz\, dr\, dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos t} r\,(4 - r\cos t)\, dr\, dt$$

$$= \int_{-\pi/2}^{\pi/2} \left[2r^2 - \frac{r^3}{3}\cos t\right]_{r=0}^{r=2\cos t} dt$$

4

$$= \int_{-\pi/2}^{\pi/2} \left(8\cos^2 t - \frac{8}{3}\cos^4 t \right) dt$$

$$= 8 \int_{-\pi/2}^{\pi/2} \cos^2 t \, dt - \frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos^4 t \, dt$$

$$*) = 8 \cdot \frac{\pi}{2} - \frac{\cancel{8}}{3} \int_{-\pi/2}^{\pi/2} \frac{3 + 4\cos 2t + \cos 4t}{\cancel{8}} \, dt$$

$$= 4\pi - \frac{1}{3} \left[3t + 2\sin 2t + \frac{\sin 4t}{4} \right]_{-\pi/2}^{\pi/2}$$

$$= 4\pi - \frac{1}{3} \left(\frac{3\pi}{2} + 0 + 0 + \frac{3\pi}{2} - 0 - 0 \right)$$

$$= 4\pi - \pi = \underline{\underline{3\pi}}$$

$$*) \quad \cos^4 t = (\cos^2 t)^2 = \left(\frac{1 + \cos 2t}{2} \right)^2$$

$$= \frac{1 + 2\cos 2t + \cos^2 2t}{4}$$

$$= \frac{1 + 2\cos 2t + \frac{1 + \cos 4t}{2}}{4}$$

$$= \frac{3 + 4\cos 2t + \cos 4t}{8}$$