Assume there is an $x \in P' \setminus P$. We choose $y \in \operatorname{int} P$ and consider $[y, x] \cap P$. Since $P$ is compact and convex (and $x \notin P$), there is a $z \in (y, x)$ with $\{z\} = [y, x] \cap \operatorname{bd} P$. By the support theorem there is a supporting hyperplane of $P$ through $z$, and hence there is a support set $F_i$ of $P$ with $z \in F_i \subset \operatorname{bd} H_i$. On the other hand, since $y \in \operatorname{int} H_i$, $x \in P' \subset H_i$, and $z \in (y, x)$, we have $z \in \operatorname{int} H_i$, a contradiction. $\square$

## *Exercises and Supplements for Sect. 1.4*

1. Let $A \subset \mathbb{R}^n$ be closed and $\operatorname{int} A \neq \emptyset$. Show that $A$ is convex if and only if every boundary point of $A$ is a support point. Does the assertion remain true without the assumption $\operatorname{int} A \neq \emptyset$?
2. Let $A \subset \mathbb{R}^n$ be closed. Suppose that for each $x \in \mathbb{R}^n$ there is a unique point $p(A, x) \in A$ such that $\|x - p(A, x)\| = \min\{\|x - y\| : y \in A\}$. Show that $A$ is convex (Motzkin's theorem).
3.* Let $A \subset \mathbb{R}^n$ be nonempty, closed and convex. Show that $A$ is compact if and only if for each $u \in \mathbb{S}^{n-1}$ there is some $\alpha \in \mathbb{R}$ such that $A \subset H^-(u, \alpha)$.
4. Let $A, B \subset \mathbb{R}^n$ be convex, $A$ closed, $B$ compact, and assume that $A \cap B = \emptyset$. Show that there is a hyperplane $\{f = \gamma\}$ and there is an $\varepsilon > 0$ such that $A \subset \{f \geq \gamma + \varepsilon\}$ and $B \subset \{f \leq \gamma - \varepsilon\}$.
5. A Bavarian farmer is the happy owner of a large herd of happy cows, consisting of totally black and totally white animals. One day he finds them sleeping in the sun in his largest meadow. Watching them, he notices that for any four cows it would be possible to build a straight fence separating the black cows from the white ones.
   Show that the farmer could build a straight fence, separating the whole herd into black and white animals.
   *Hint:* Cows are lazy. When they sleep, they sleep—even if you build a fence across the meadow. Warning: Cows are not convex and certainly they are not points.
6. Let $F_1, \ldots, F_m$ be the facets of an $n$-dimensional polytope $P \subset \mathbb{R}^n$, and let $H_1, \ldots, H_m$ be the corresponding supporting halfspaces containing $P$. Show that

$$P = \bigcap_{i=1}^{m} H_i. \qquad (*)$$

   (This is a generalization of the representation shown in the proof of Theorem 1.20.) Show further that the representation $(*)$ is minimal in the sense that, for each representation

$$P = \bigcap_{i \in I} \widetilde{H}_i,$$

   with a family of halfspaces $\{\widetilde{H}_i : i \in I\}$, we have $\{H_1, \ldots, H_m\} \subset \{\widetilde{H}_i : i \in I\}$.

7. Let $A_1, \ldots, A_{n+1} \subset \mathbb{R}^n$ be sets. Suppose that $x \in \bigcap_{i=1}^{n+1} \operatorname{conv}(A_i)$. Then there are points $a_i \in A_i$ for $i = 1, \ldots, n+1$ such that $x \in \operatorname{conv}\{a_1, \ldots, a_{n+1}\}$.

   **Interpretation** The points of the set $A_i$ are assigned the colour $i$. Then the result says that if a point lies in the convex hull of the points having colour $i$, for $i = 1, \ldots, n+1$, then the point is in the convex hull of a colourful simplex the vertices of which show the $n+1$ different colours. In the special case of equal sets $A_1 = \ldots = A_{n+1} =: A$ (all the points of $A$ exhibit all $n+1$ colours), the assertion follows from Carathéodory's theorem. For this reason, the assertion is a colourful version of Carathéodory's theorem.

8.* Let $A \subset \mathbb{R}^n$ be closed and convex. Show that the intersection of a family of support sets of $A$ is a support set of $A$ or the empty set.

9. Let $K_1, \ldots, K_m \subset \mathbb{R}^n$ be compact convex sets with $\bigcap_{i=1}^m K_i = \emptyset$. Show that there are closed halfspaces $H_1^+, \ldots, H_m^+ \subset \mathbb{R}^n$ such that $K_i \subset H_i^+$ for $i = 1, \ldots, m$ and such that $\bigcap_{i=1}^m H_i^+ = \emptyset$.

10. Let $A := \{(0, y, 1)^\top \in \mathbb{R}^3 : y \in \mathbb{R}\}$ and $B := \{(x, y, z)^\top \in \mathbb{R}^3 : x, y, z \geq 0, xy \geq z^2\}$. Show that $A, B$ are disjoint, closed, convex sets. Determine the distance of the sets, that is,

$$d(A, B) := \inf\{\|a - b\| : a \in A, b \in B\}.$$

    Determine all hyperplanes separating $A$ and $B$.

11.* For $A \subset \mathbb{R}^n$ and $a \in A$, let

$$N(A, a) := \{u \in \mathbb{R}^n : \langle u, x - a\rangle \leq 0 \text{ for } x \in A\}.$$

    Prove the following assertions.

    (a) $N(A, a)$ is a closed convex cone and $N(A, a) = N(\operatorname{cl} A, a)$. (A set $C \subset \mathbb{R}^n$ is called a cone if $\lambda C \subset C$ for $\lambda \geq 0$.)
    (b) If $A$ is convex, then $N(A, a) = N(A \cap B^n(a, \varepsilon), a)$, for $\varepsilon > 0$.
    (c) If $A$ is convex and $a \in \operatorname{bd} A$, then $\dim N(A, a) \geq 1$, whereas $N(A, a) = \{0\}$ if $a \in \operatorname{int} A$.

    The set $N(A, a)$ is called the *normal cone* of $A$ at $a$.

12.* Let $A, B \subset \mathbb{R}^n$, $a \in A$ and $b \in B$. Show that

$$N(A + B, a + b) = N(A, a) \cap N(B, b).$$

13.* Let $A, B \subset \mathbb{R}^n$ be convex sets and $c \in A \cap B$. Suppose that $\operatorname{relint}(A) \cap \operatorname{relint}(B) \neq \emptyset$. Show that

$$N(A \cap B, c) = N(A, c) + N(B, c).$$

    In particular, the sum $N(A, c) + N(B, c)$ is closed.