Předmět: NMTM101 Matematická analýza I

Typ výuky: Cvičení

Vahid Borji
borji@karlin.mff.cuni.cz

$$2.\,b)\ \Big|\underbrace{|x-3|-2}\Big| = 1 \iff \begin{cases} |x-3|-2=1 \\ \vee \\ |x-3|-2=-1 \end{cases}$$

$$|x-3|-2=1 \Rightarrow |x-3|=3 \Rightarrow \begin{cases} x-3=3 \rightarrow x=6 \\ x-3=-3 \rightarrow x=0 \end{cases}$$

$$|x-3|-2=-1 \Rightarrow |x-3|=1 \Rightarrow \begin{cases} x-3=1 \rightarrow x=4 \\ x-3=-1 \rightarrow x=2 \end{cases}$$

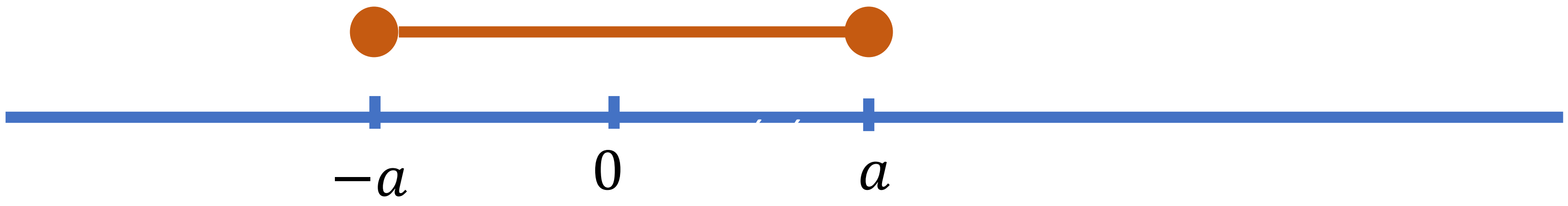
$$x \in \{0,2,4,6\}$$

$\mathbf{2.}\,\boldsymbol{c})\ \left| |\boldsymbol{x}-\mathbf{2}| + \mathbf{1} \right| \le \mathbf{5}$

Pamatování

$$|\boldsymbol{x}| \le \boldsymbol{a} \Rightarrow |\boldsymbol{x}-\mathbf{0}| \le \boldsymbol{a}$$

vzdálenost mezi x a nulou je menší nebo rovna a

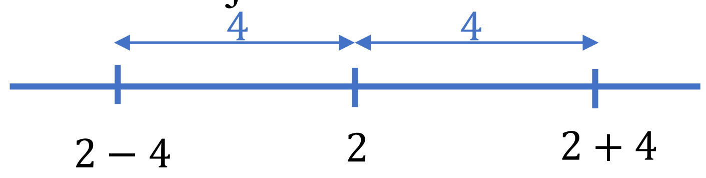


$$-\boldsymbol{a} \le \boldsymbol{x} \le \boldsymbol{a}$$

$\mathbf{2.}\,\boldsymbol{c})\ \left|\underbrace{|\boldsymbol{x}-\mathbf{2}|+\mathbf{1}}\right| \le \mathbf{5}$ vzdálenost mezi $|x-2|+1$ a nulou je menší nebo rovna 5

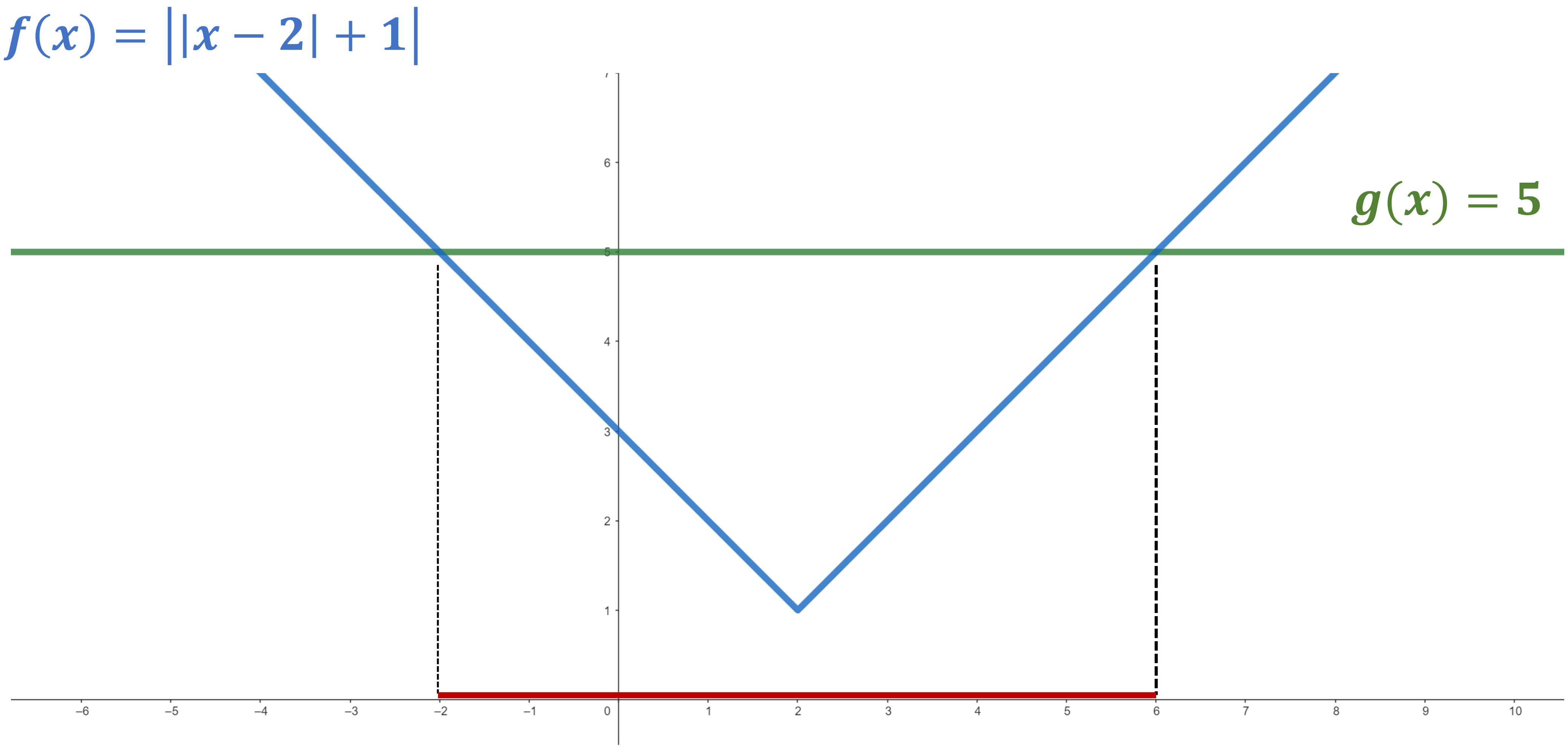
$\mathbf{-5 \le |\boldsymbol{x}-2|+1 \le 5} \quad \Rightarrow \mathbf{-6 \le |\boldsymbol{x}-2| \le 4} \quad \Rightarrow \underbrace{|\boldsymbol{x}-\mathbf{2}|} \le \mathbf{4}$

vzdálenost mezi x a 2 je menší nebo rovna 4



$\Rightarrow \mathbf{-4 \le \boldsymbol{x}-2 \le 4} \qquad \Rightarrow \mathbf{-2 \le \boldsymbol{x} \le 6}$

$2. c)\ \left||x-2|+1\right| \leq 5$

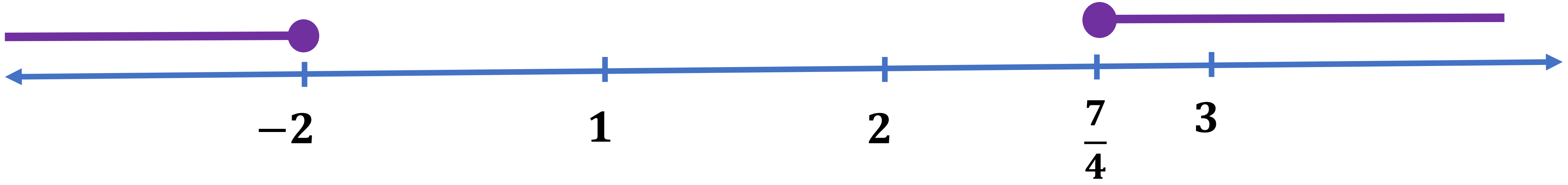


$x \in [-2, 6]$

$$2.\, d)\ \left|x^2 - 4x + 3\right| \leq |x^2 - 4|$$

$$|(x-3)(x-1)| \leq |(x-2)(x+2)|$$

$\boxed{x = 3, \quad x = 1, \quad x = 2, \quad x = -2}$



$\boxed{x \leq -2:} \Rightarrow \cancel{x^2} - 4x + 3 \leq \cancel{x^2} - 4 \Rightarrow -4x + 3 \leq -4 \Rightarrow -4x \leq -7 \Rightarrow x \geq \dfrac{7}{4}$

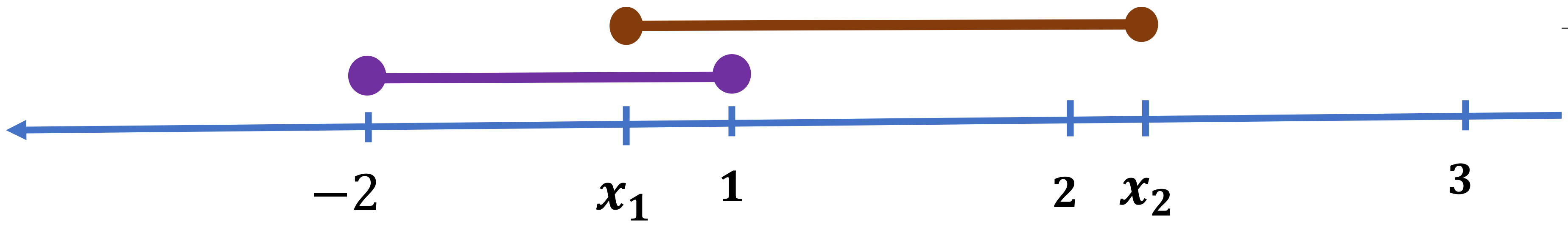
$$x \leq -2 \quad \wedge \quad x \geq \frac{7}{4}$$

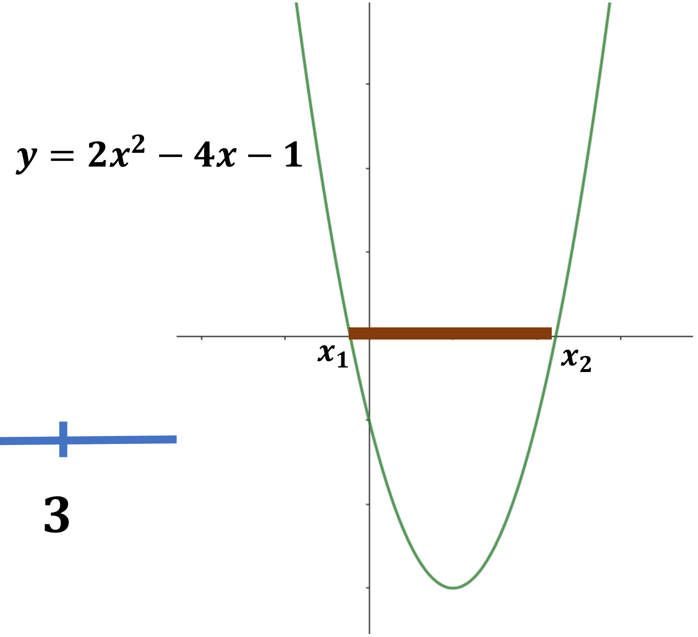
$$\emptyset$$

V tomto intervalu nemáme řešení

$$2.\,d)\ |x^2 - 4x + 3| \leq |x^2 - 4|$$

$$|(x-3)(x-1)| \leq |(x-2)(x+2)|$$





$\boxed{-2 \leq x \leq 1:}$ $\Rightarrow x^2 - 4x + 3 \leq -x^2 + 4 \Rightarrow 2x^2 - 4x - 1 \leq 0 \Rightarrow 2x^2 - 4x - 1 = 0$

$\boxed{a = 2,\ b = -4,\ c = -1}$

$$x_1 = \frac{4 - \sqrt{24}}{4} = \frac{4 - 2\sqrt{6}}{4} = 1 - \frac{\sqrt{6}}{2} \approx -0.225$$

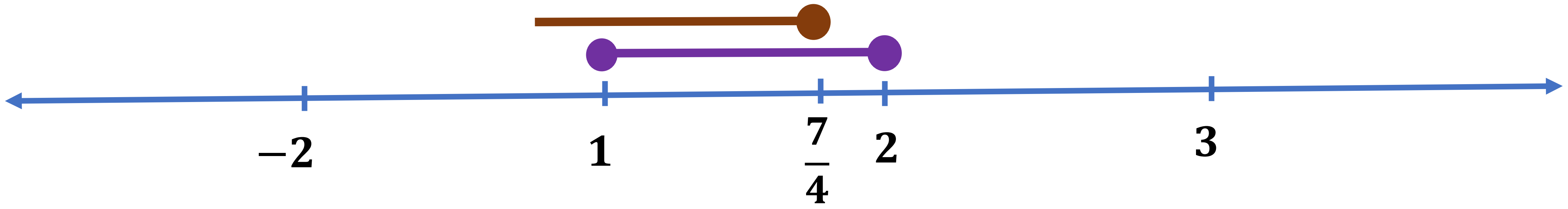
$$\boxed{x_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}}$$

$$x_2 = \frac{4 + \sqrt{24}}{4} = \frac{4 + 2\sqrt{6}}{4} = 1 + \frac{\sqrt{6}}{2} \approx 2.225$$

$$x \in [1 - \frac{\sqrt{6}}{2}, 1 + \frac{\sqrt{6}}{2}] \wedge x \in [-2, 1] \Rightarrow \boxed{x \in [1 - \frac{\sqrt{6}}{2}, 1]}$$

$$2.\ d)\ \left|x^2 - 4x + 3\right| \leq |x^2 - 4|$$

$$|(x-3)(x-1)| \leq |(x-2)(x+2)|$$



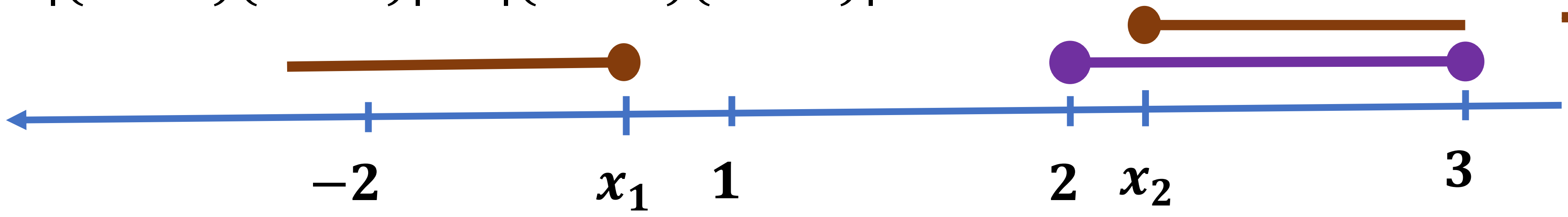
$\boxed{1 \leq x \leq 2:}$ $\Rightarrow -x^2 + 4x - 3 \leq -x^2 + 4 \Rightarrow 4x \leq 7 \Rightarrow x \leq \dfrac{7}{4}$

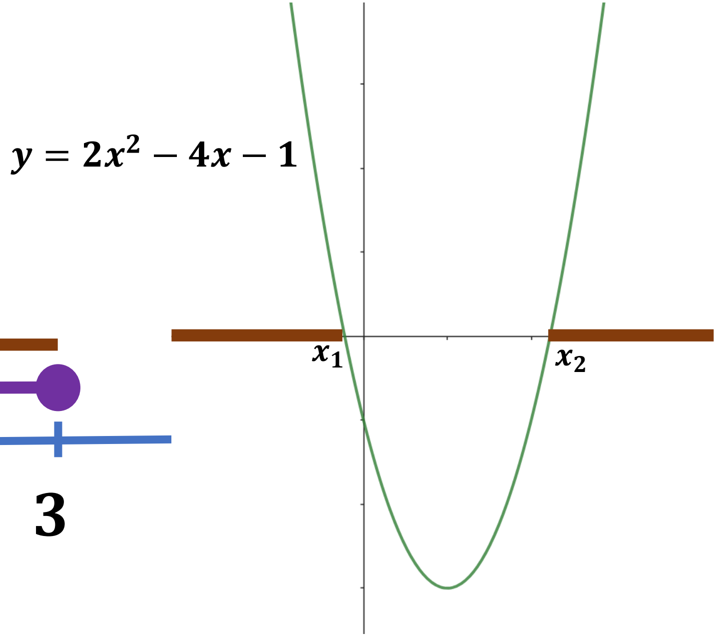
$$(1 \leq x \leq 2) \wedge \left(x \leq \frac{7}{4}\right)$$

$$\boxed{1 \leq x \leq \frac{7}{4}}$$

$$2.\ d)\ \left|x^2 - 4x + 3\right| \le \left|x^2 - 4\right|$$

$$|(x-3)(x-1)| \le |(x-2)(x+2)|$$





$\boxed{2 \le x \le 3:}$ $\quad -x^2 + 4x - 3 \le x^2 - 4 \Rightarrow 2x^2 - 4x - 1 \ge 0 \Rightarrow 2x^2 - 4x - 1 = 0$

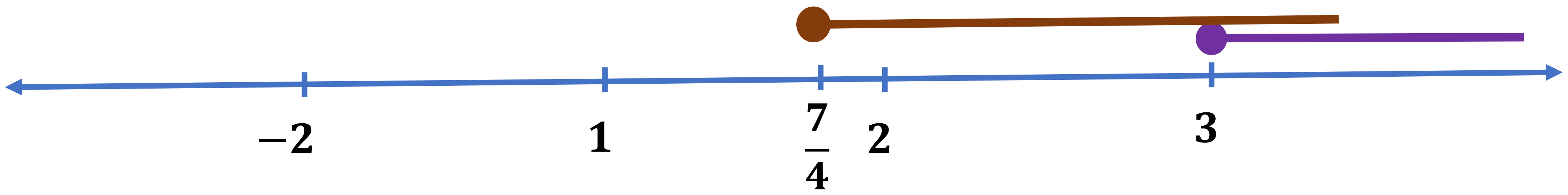
$$x_1 = \frac{4 - \sqrt{24}}{4} = \frac{4 - 2\sqrt{6}}{4} = 1 - \frac{\sqrt{6}}{2} \approx -0.225$$

$$x_2 = \frac{4 + \sqrt{24}}{4} = \frac{4 + 2\sqrt{6}}{4} = 1 + \frac{\sqrt{6}}{2} \approx 2.225$$

$$\left(x \le 1 - \frac{\sqrt{6}}{2} \vee x \ge 1 + \frac{\sqrt{6}}{2}\right) \wedge \ (2 \le x \le 3) \implies \boxed{x \in \left[1 + \frac{\sqrt{6}}{2}, 3\right]}$$

$2.\,d)\ \left|x^2 - 4x + 3\right| \le \left|x^2 - 4\right|$

$|(x-3)(x-1)| \le |(x-2)(x+2)|$



$\boxed{x \ge 3:}$ $\Rightarrow \cancel{x^2} - 4x + 3 \le \cancel{x^2} - 4 \Rightarrow -4x + 3 \le -4 \Rightarrow -4x \le -7 \Rightarrow x \ge \frac{7}{4}$

$$x \ge 3 \quad \wedge \quad x \ge \frac{7}{4}$$

$$\boxed{x \ge 3}$$

$$2.\, d)\ \left|x^2 - 4x + 3\right| \leq |x^2 - 4|$$

$$\boxed{1 - \frac{\sqrt{6}}{2} \leq x \leq 1} \lor \boxed{1 \leq x \leq \frac{7}{4}} \lor \boxed{1 + \frac{\sqrt{6}}{2} \leq x \leq 3} \lor \boxed{x \geq 3}$$

$$\boxed{1 - \frac{\sqrt{6}}{2} \leq x \leq \frac{7}{4} \lor 1 + \frac{\sqrt{6}}{2} \leq x}$$

$$2.\, d)\ \left|x^2 - 4x + 3\right| \le |x^2 - 4|$$

$$y = \left|x^2 - 4x + 3\right|$$

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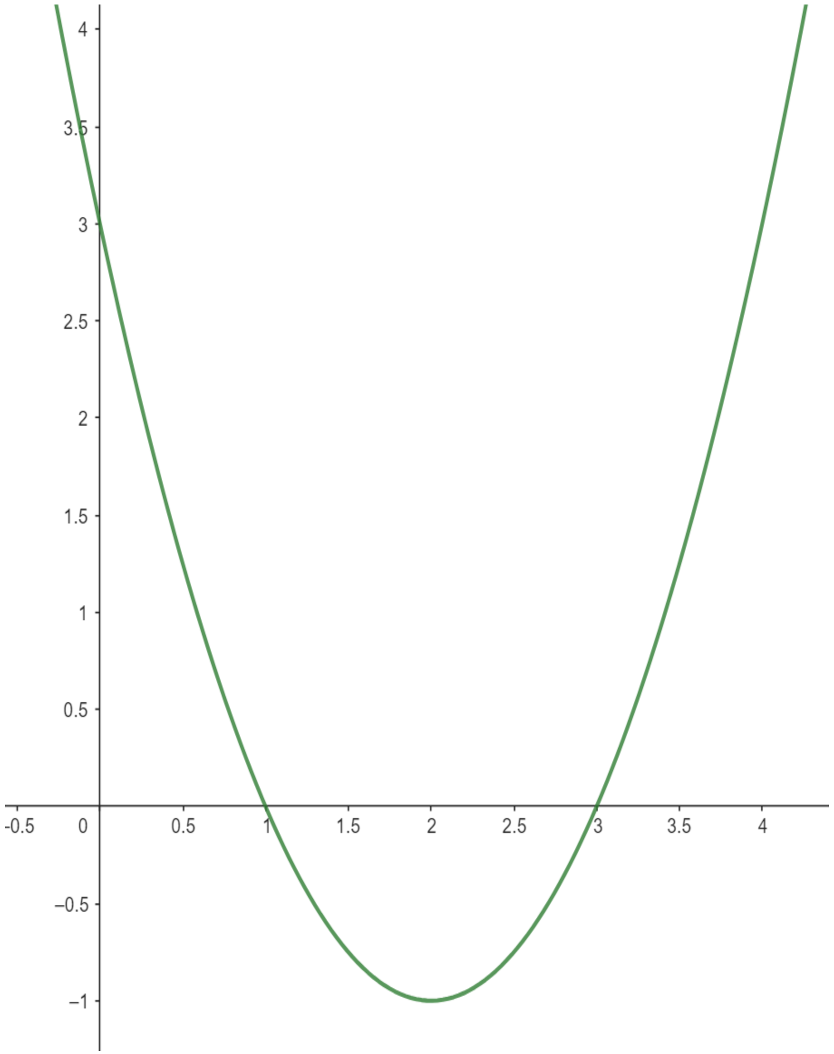
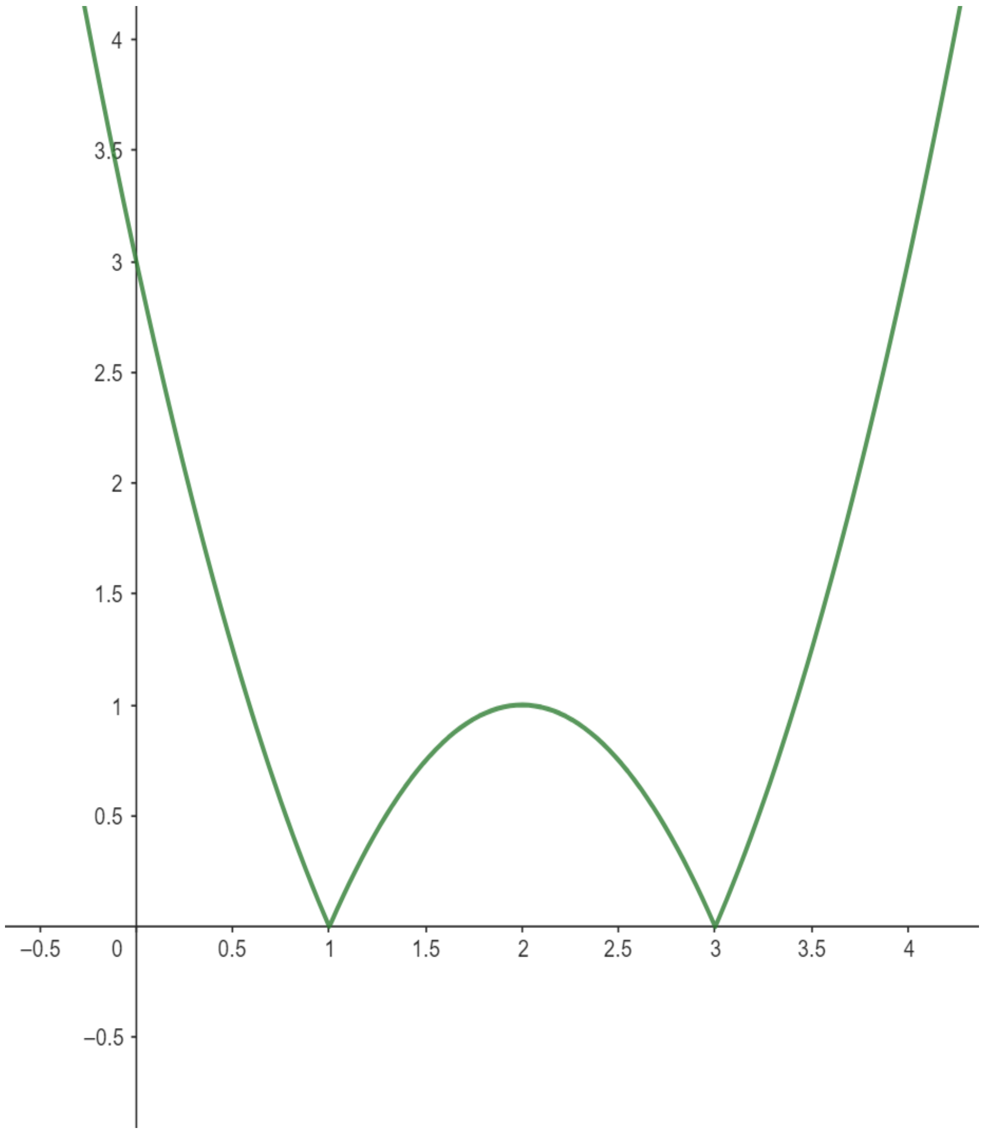
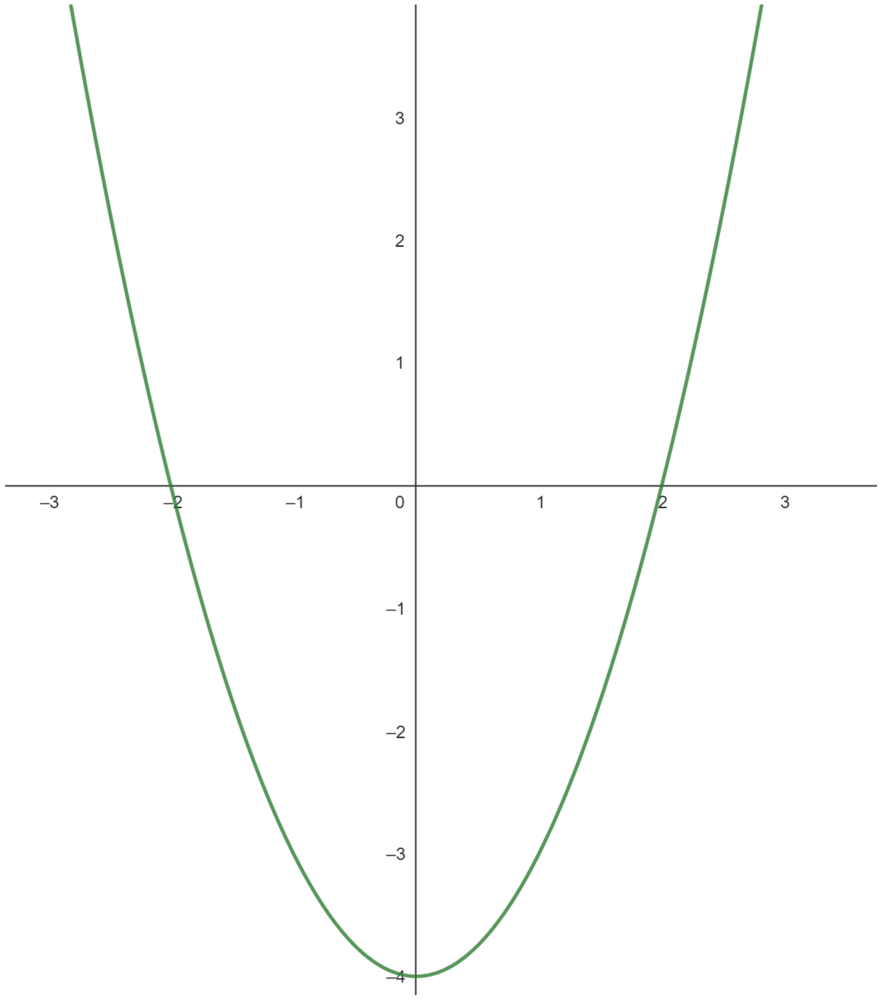


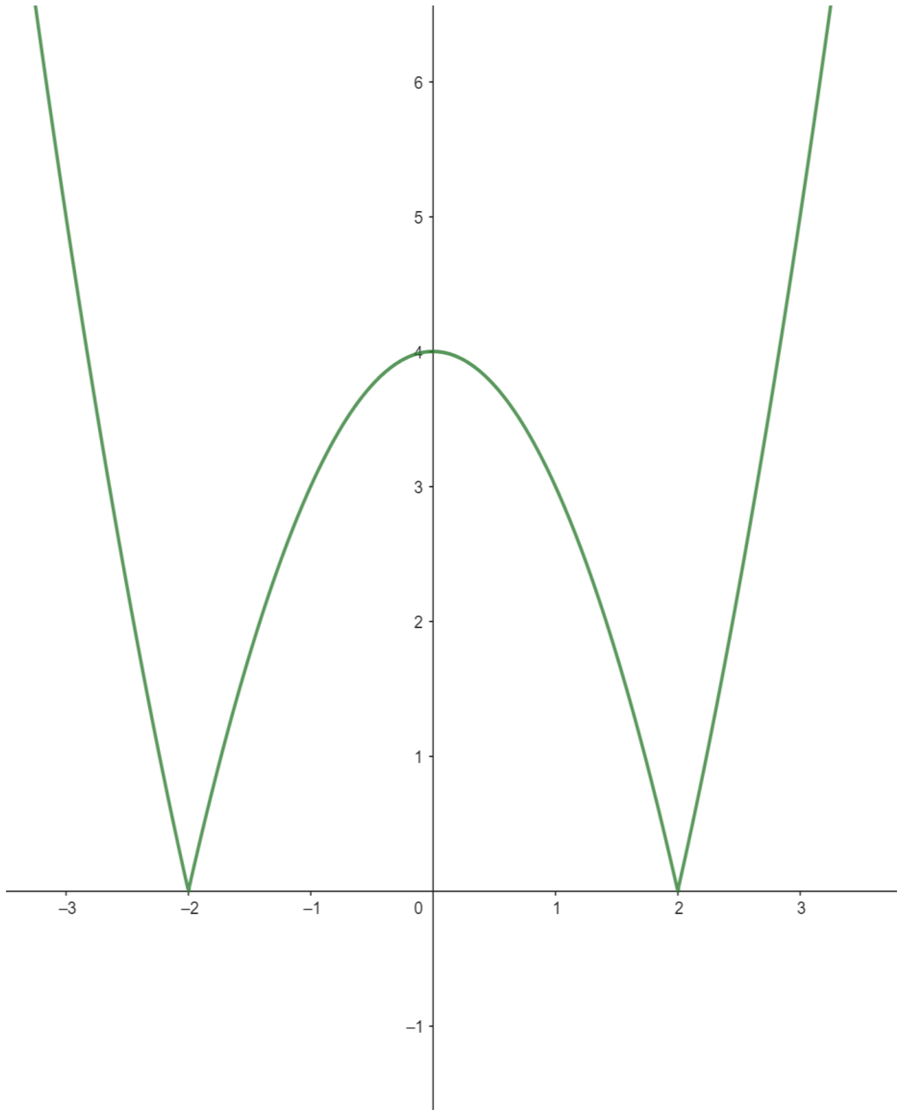


$2.\, d)\ \left|x^2 - 4x + 3\right| \leq |x^2 - 4|$

$y = x^2 - 4$



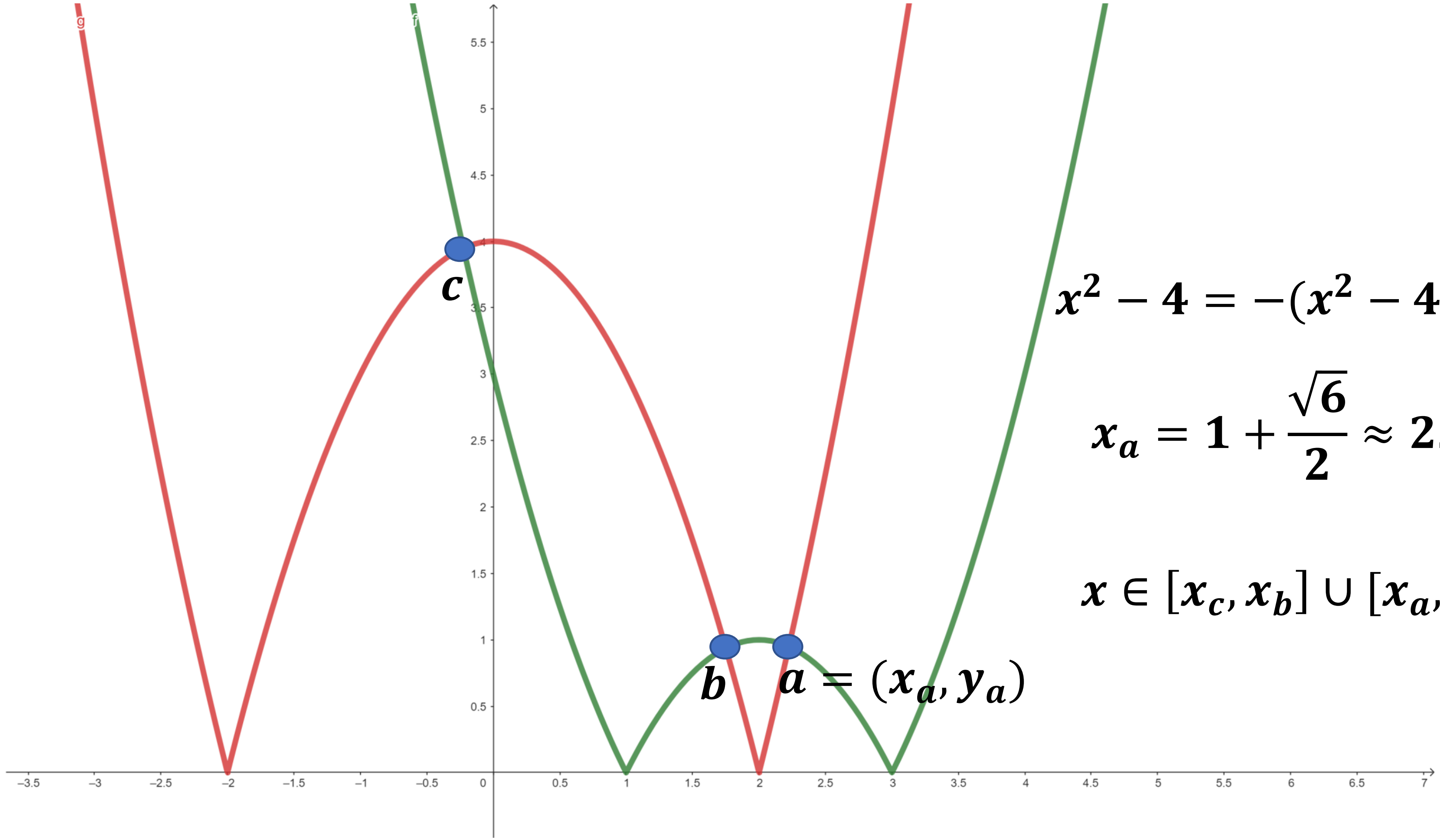
$y = \left|x^2 - 4\right|$



2. d) $\left|x^2-4x+3\right| \leq \left|x^2-4\right|$

$y=\left|x^2-4x+3\right|$

$y=\left|x^2-4\right|$

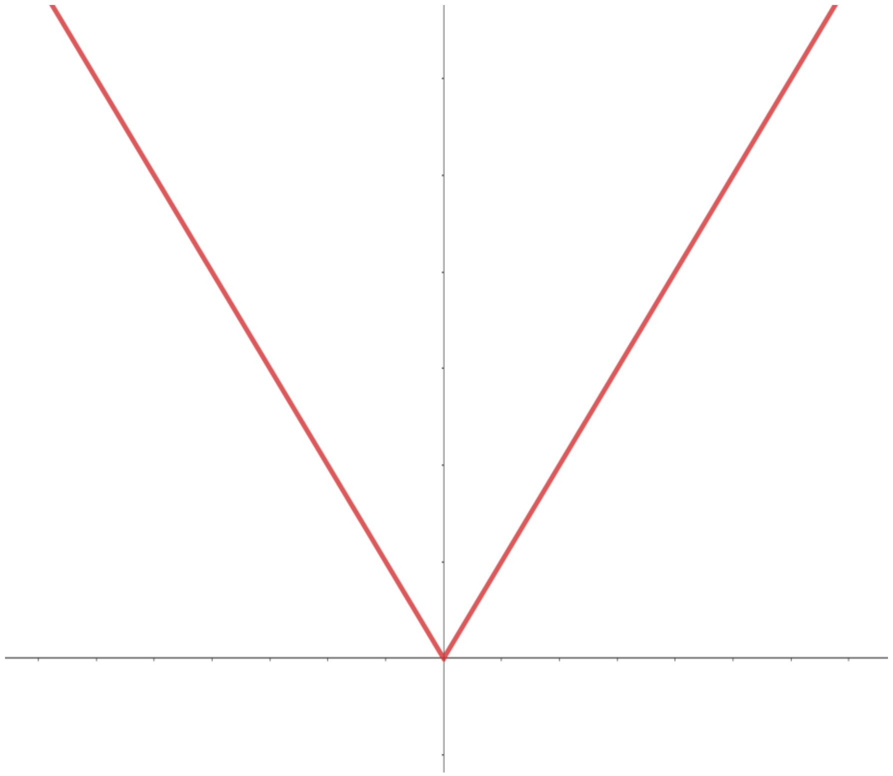


$$x^2-4=-(x^2-4x+3)$$

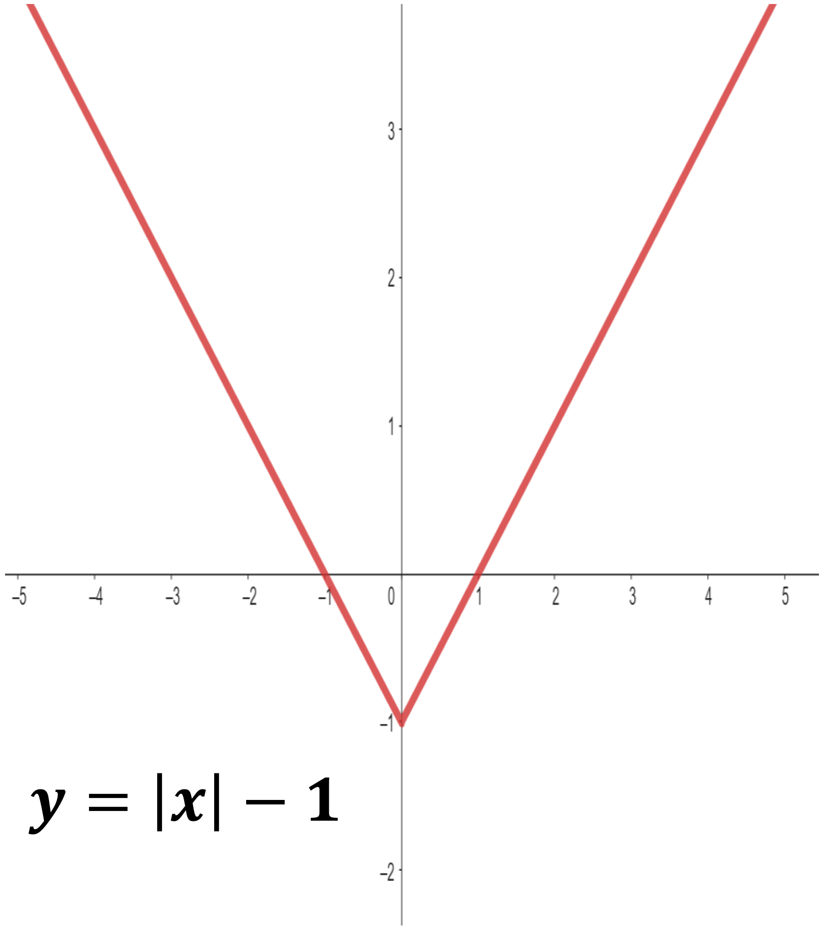
$$x_a=1+\frac{\sqrt{6}}{2}\approx 2.225$$

$$x\in[x_c,x_b]\cup[x_a,\infty)$$

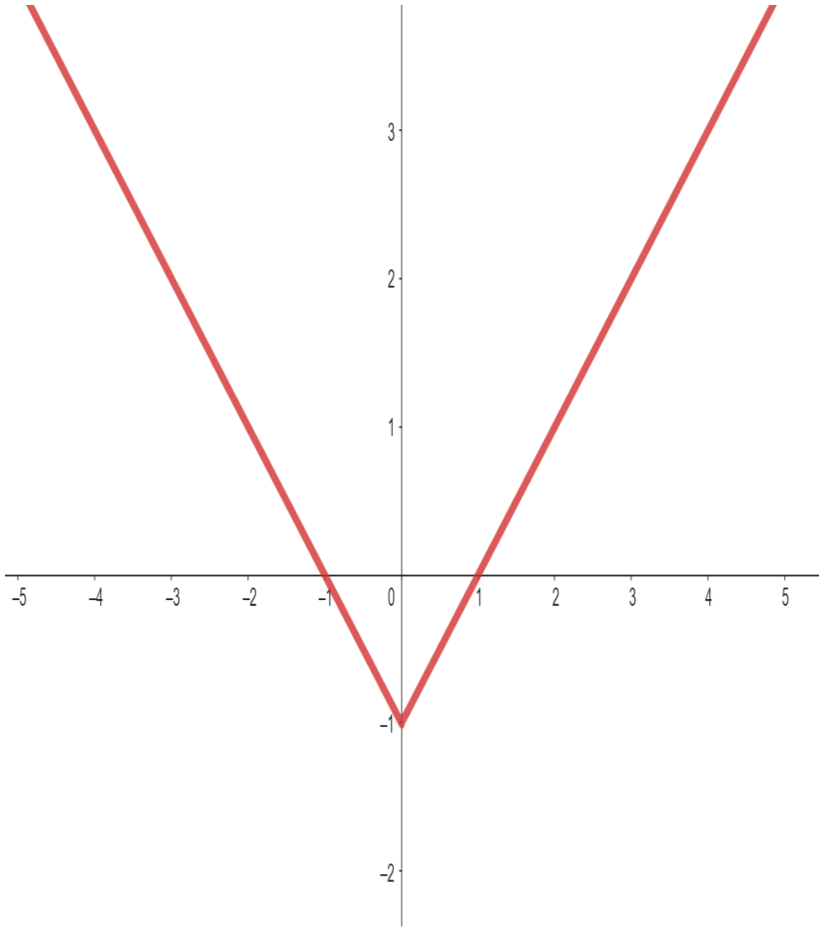
$$2.\ e)\ y = ||||x| - 1| - 1| - 1|$$



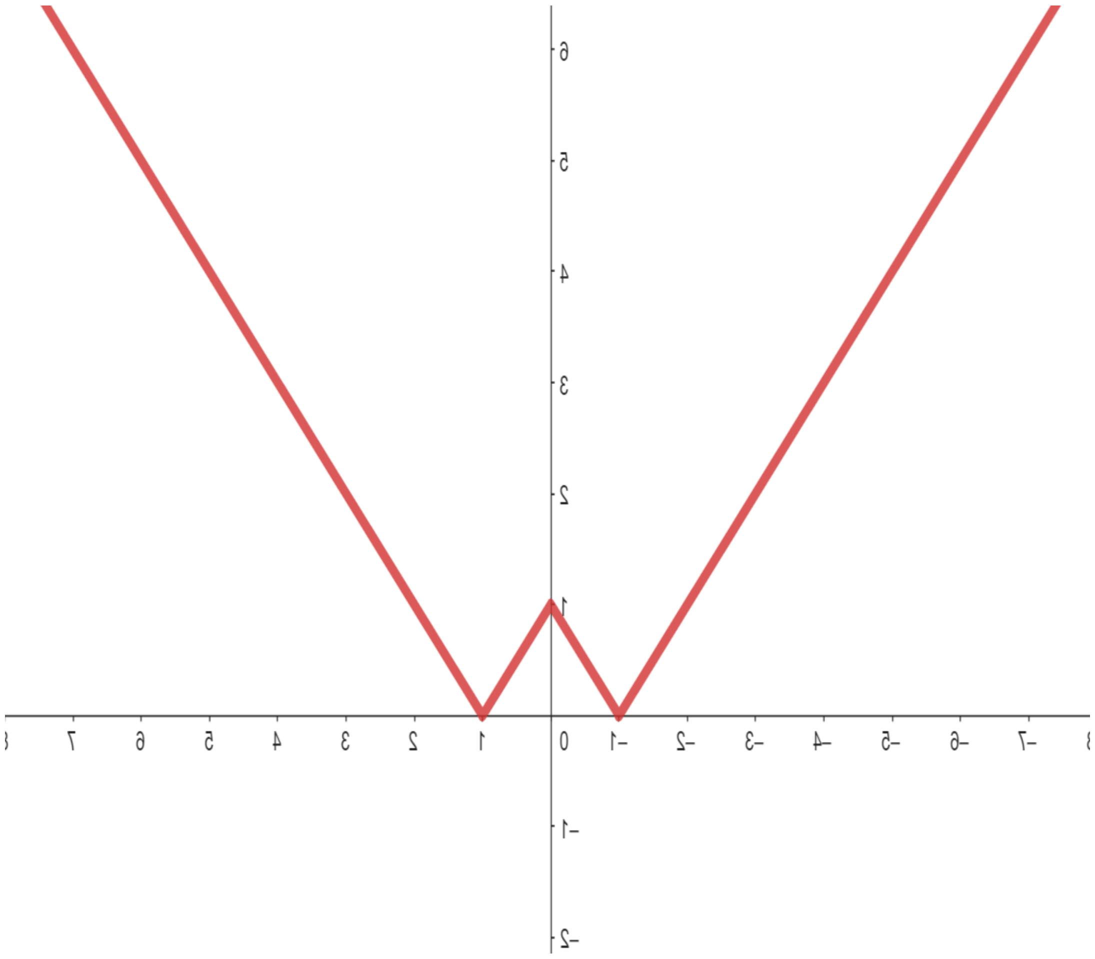
$$y = |x|$$



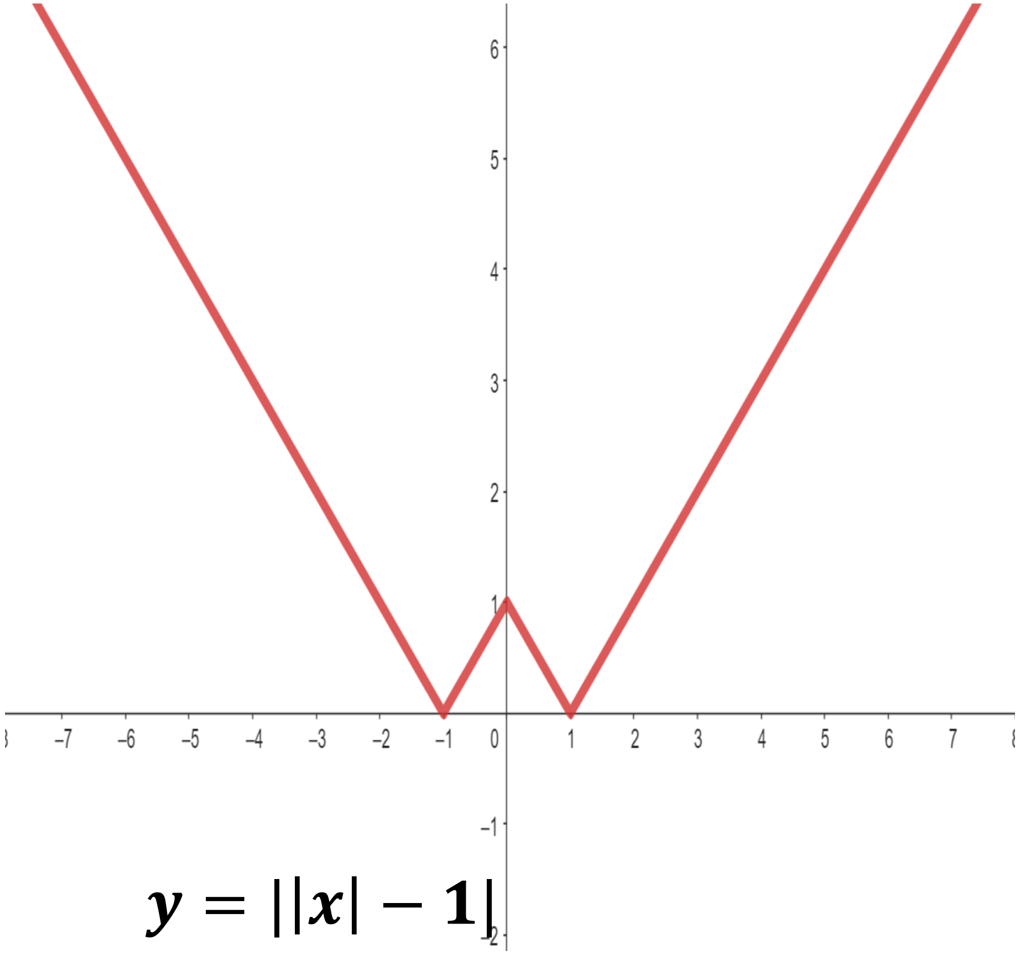
$$y = |x| - 1$$



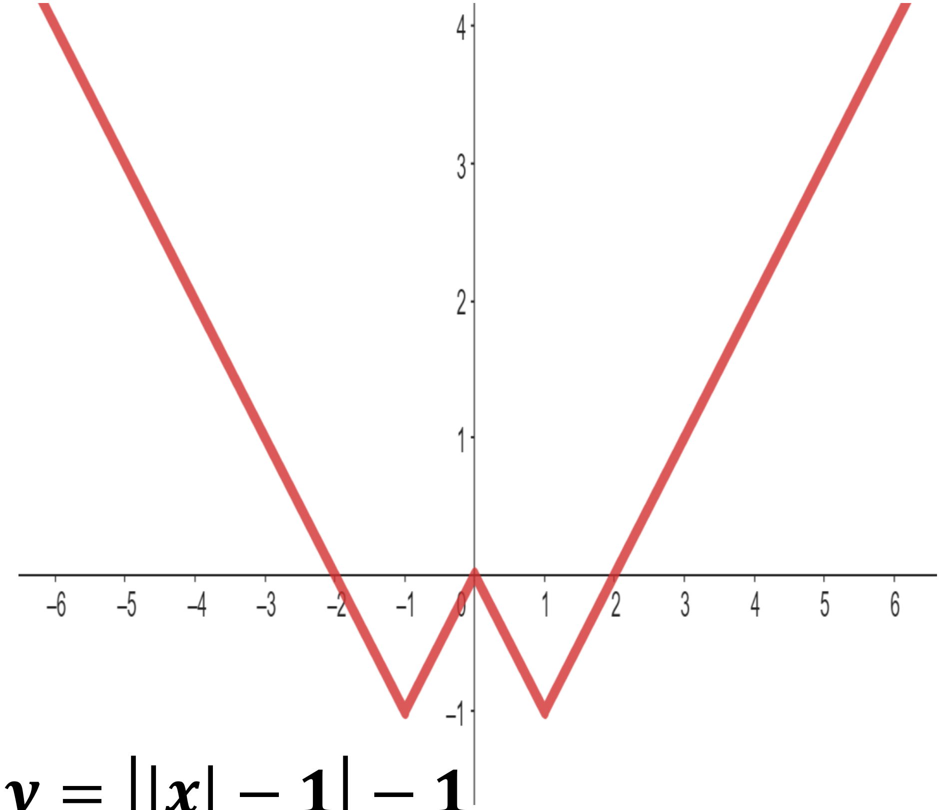
$y = |x| - 1$



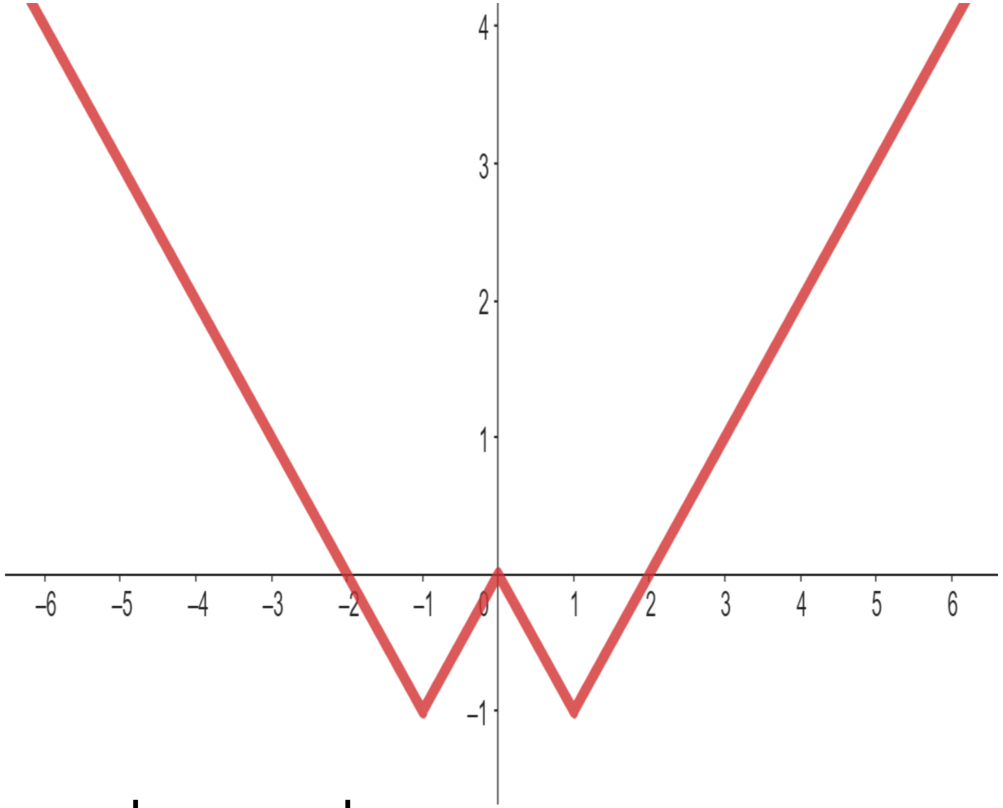
$y = ||x| - 1|$



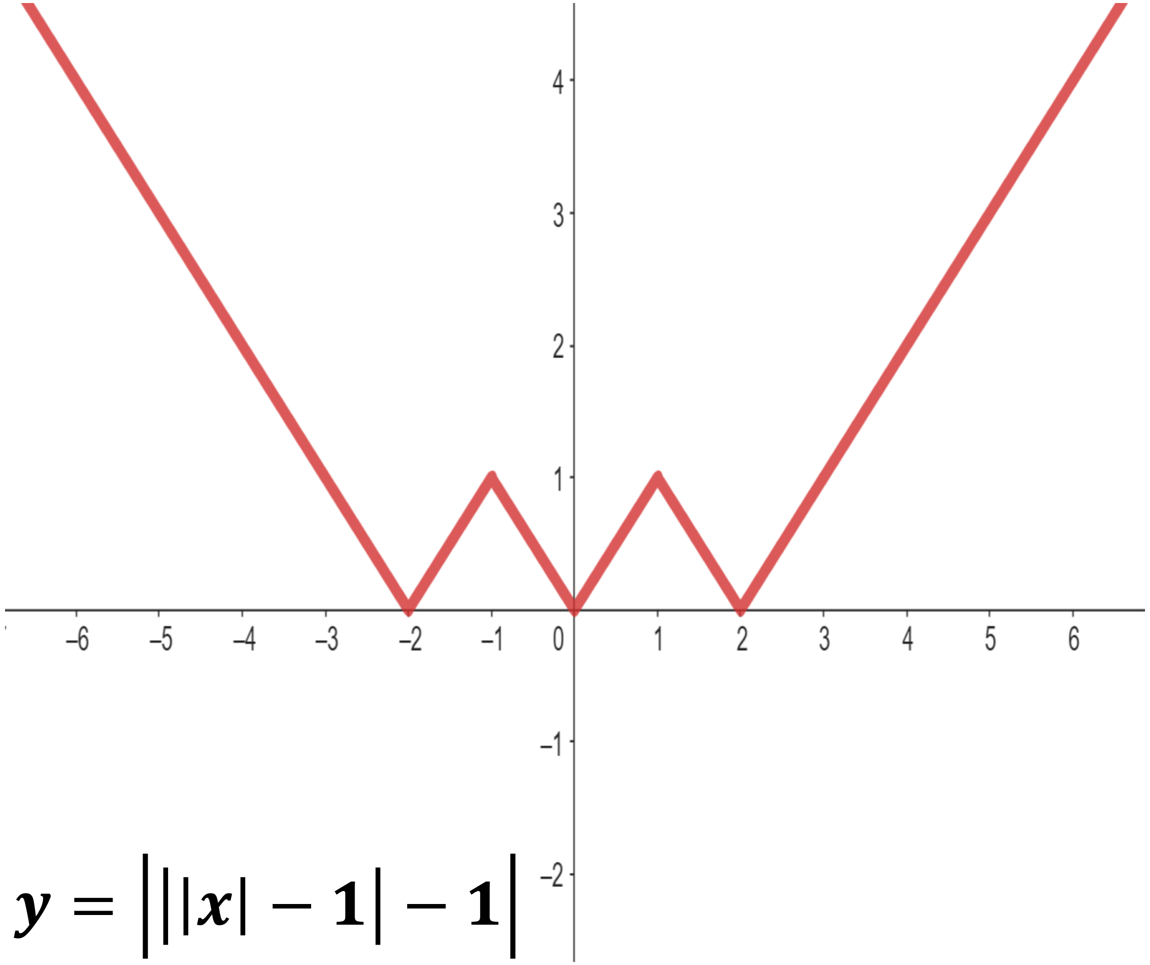
$$y = \big||x| - 1\big|$$



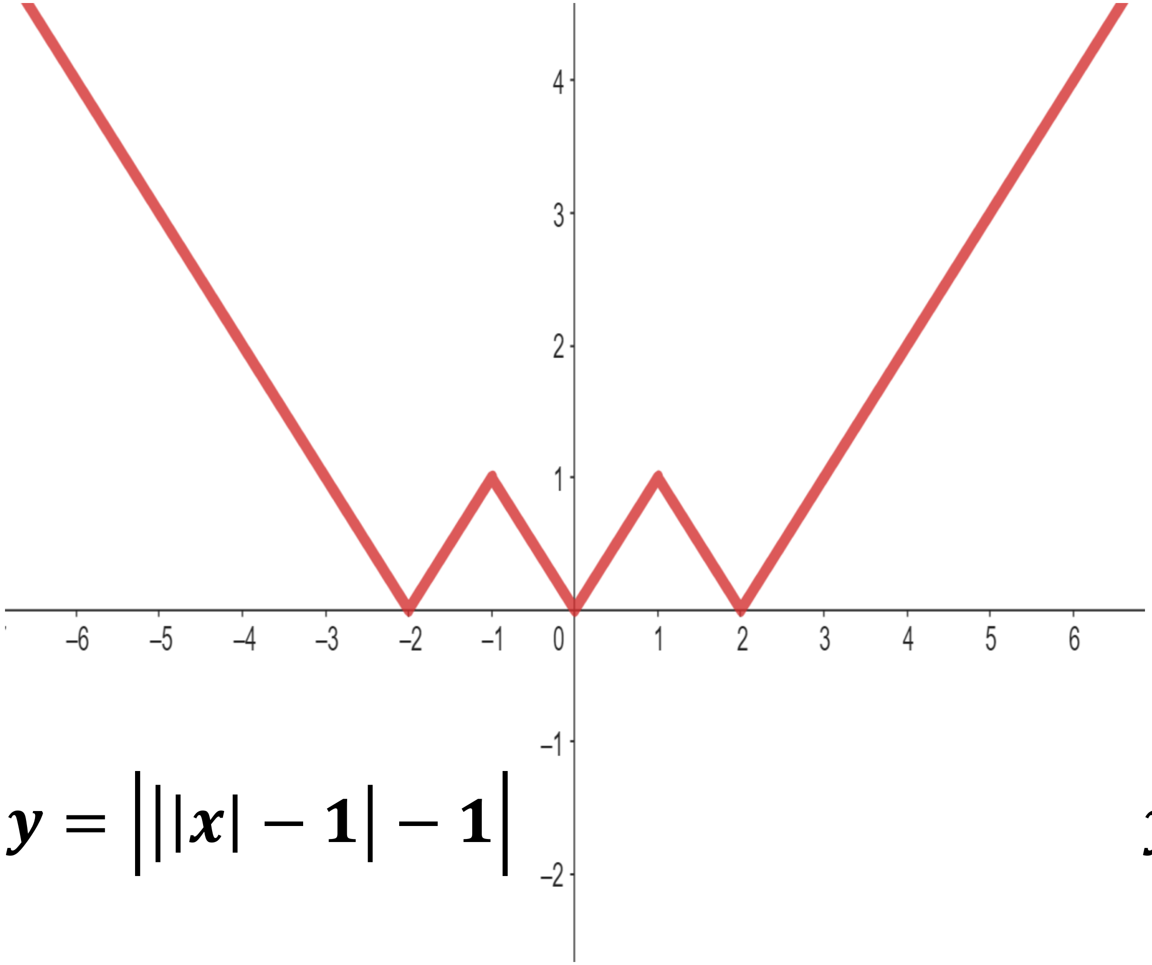
$$y = \big||x| - 1\big| - 1$$



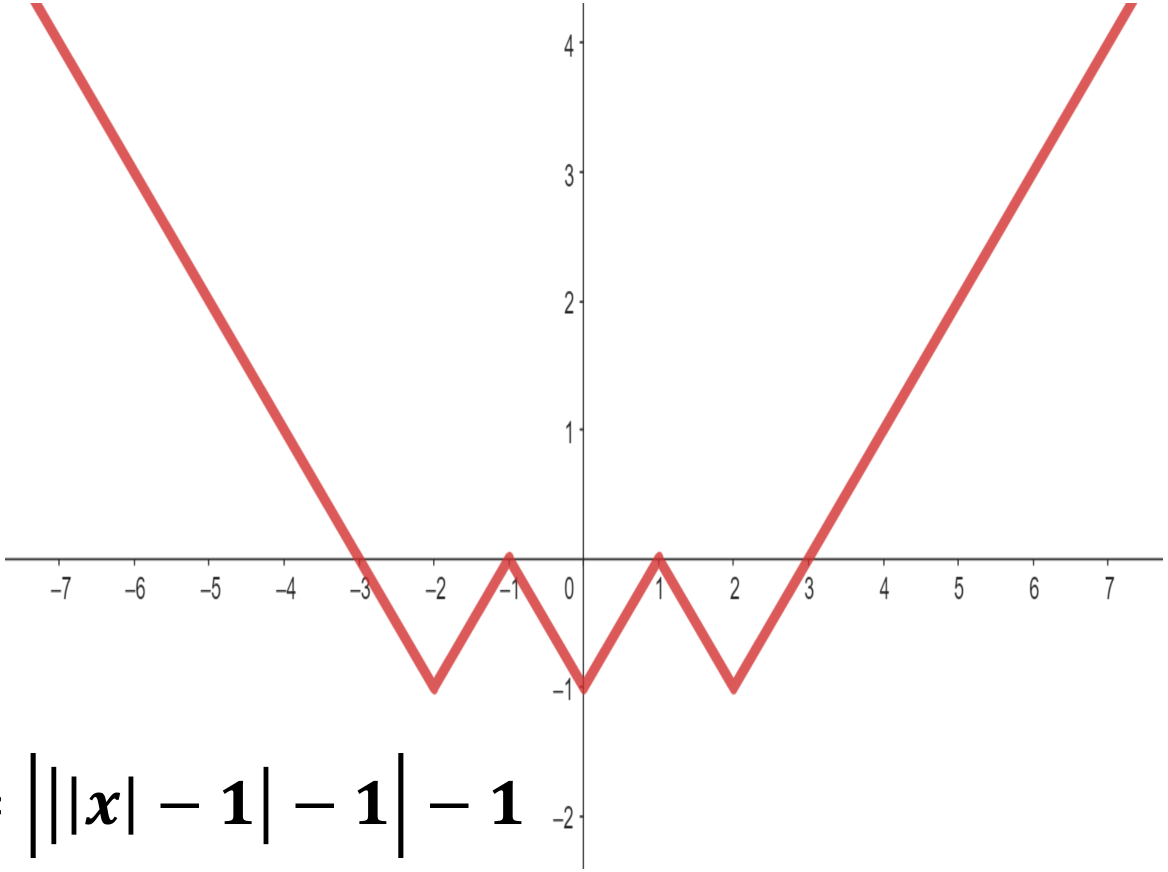
$$y = \big||x| - 1\big| - 1$$



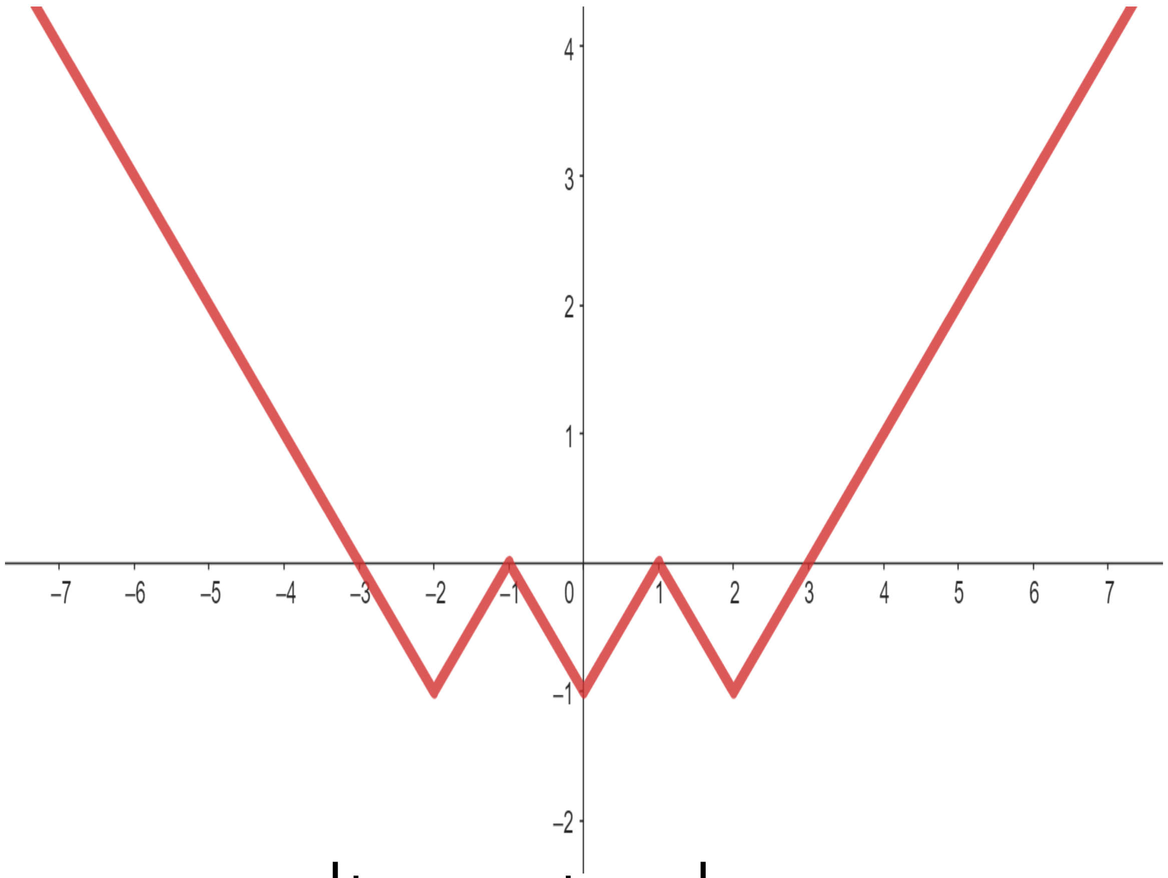
$$y = \Big|\big||x| - 1\big| - 1\Big|$$



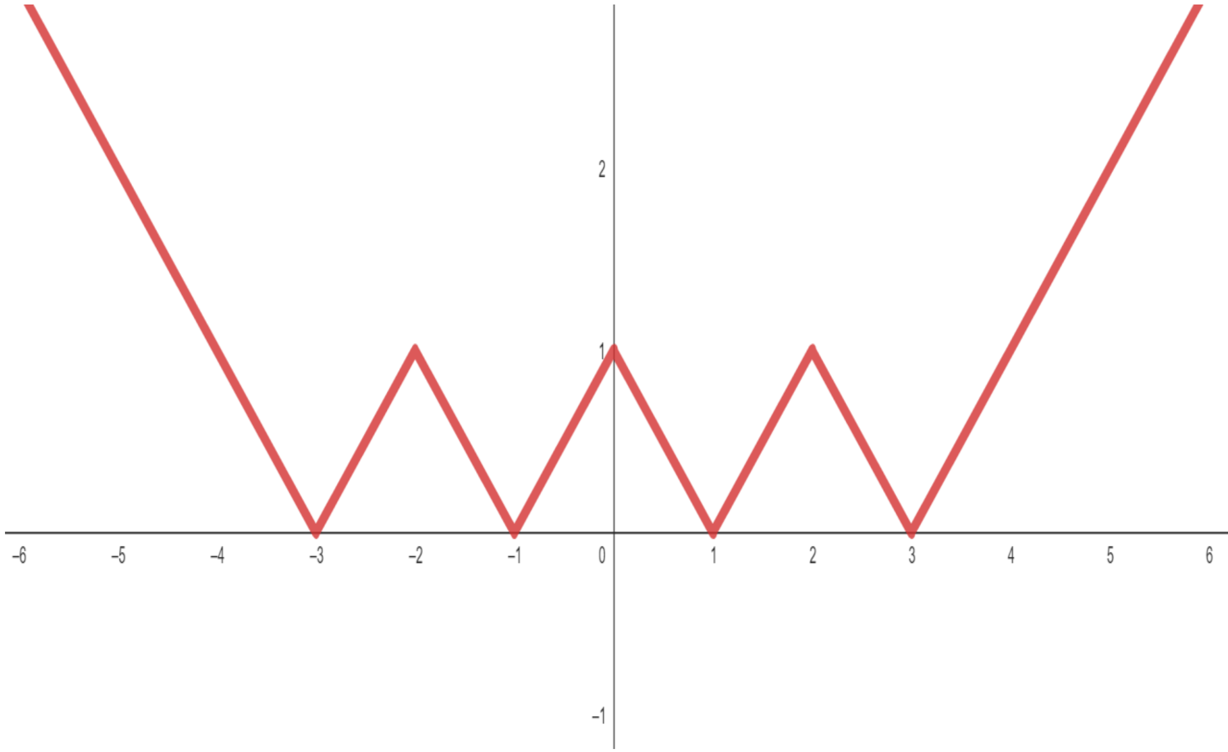
$$y = \big||x| - 1| - 1\big|$$



$$y = \big||x| - 1| - 1\big| - 1$$



$$y = \big| \big| |x| - 1 \big| - 1 \big| - 1$$



$$y = \Big| \big| \big| |x| - 1 \big| - 1 \big| - 1 \Big|$$

$\mathbf{2.}\ \boldsymbol{f})\ \forall \boldsymbol{a} \in \mathbb{R}\ \forall \boldsymbol{b} \in \mathbb{R}\colon |\boldsymbol{a}+\boldsymbol{b}| \le |\boldsymbol{a}| + |\boldsymbol{b}|.$

$$\boxed{|\boldsymbol{x}|^2 = \boldsymbol{x}^2}$$

$$\boxed{|\boldsymbol{x}| \ge \boldsymbol{x}}$$

$$\forall \boldsymbol{a} \in \mathbb{R}\ \forall \boldsymbol{b} \in \mathbb{R}\colon (|\boldsymbol{a}| + |\boldsymbol{b}|)^2 = |\boldsymbol{a}|^2 + 2|\boldsymbol{a}||\boldsymbol{b}| + |\boldsymbol{b}|^2 \ge \boldsymbol{a}^2 + 2\boldsymbol{ab} + \boldsymbol{b}^2 = (\boldsymbol{a}+\boldsymbol{b})^2 = |\boldsymbol{a}+\boldsymbol{b}|^2$$

$$(|\boldsymbol{a}| + |\boldsymbol{b}|)^2 \ge |\boldsymbol{a}+\boldsymbol{b}|^2$$

$$\sqrt{(|\boldsymbol{a}| + |\boldsymbol{b}|)^2} \ge \sqrt{|\boldsymbol{a}+\boldsymbol{b}|^2}$$

$$|\boldsymbol{a}| + |\boldsymbol{b}| \ge |\boldsymbol{a}+\boldsymbol{b}|$$

$$2.\, g)\ \{x \in \mathbb{R}: \Big|\underbrace{\big||x|-1\big|-2\big|-3}\Big| < 1\}$$

$$-1 < \big|\,||x|-1|-2\big| - 3 < 1$$

$$2 < \big|\,||x|-1|-2\big| < 4$$

$$(\big|\,||x|-1|-2\big| > 2) \wedge (\big|\,||x|-1|-2\big| < 4)$$

$$\big|\,||x|-1|-2\big| > 2 \Rightarrow ||x|-1|-2 > 2 \vee ||x|-1|-2 < -2 \Rightarrow ||x|-1| > 4 \vee ||x|-1| < 0$$

$$\Rightarrow ||x|-1| > 4 \quad \Rightarrow |x|-1 > 4 \vee |x|-1 < -4 \quad \Rightarrow |x| > 5 \vee |x| < -3 \Rightarrow |x| > 5$$

$$\Rightarrow x > 5 \vee x < -5$$

$$\mathbf{2.}\ \boldsymbol{g})\ \{\boldsymbol{x} \in \mathbb{R} : \Big|\big|\,|\boldsymbol{x}| - \mathbf{1}\big| - \mathbf{2}\Big| - \mathbf{3}\Big| < \mathbf{1}\}$$

$$-1 < \Big|\big||x| - 1\big| - 2\Big| - 3 < 1$$

$$2 < \Big|\big||x| - 1\big| - 2\Big| < 4$$

$$\left(\Big|\big||x| - 1\big| - 2\Big| > 2\right) \land \left(\Big|\big||x| - 1\big| - 2\Big| < 4\right)$$

$$\Big|\big||x| - 1\big| - 2\Big| < 4 \Rightarrow \big||x| - 1\big| - 2 < 4 \land \big||x| - 1\big| - 2 > -4 \Rightarrow \big||x| - 1\big| < 6 \land \big||x| - 1\big| > -2$$

$$\Rightarrow (-6 < |x| - 1 < 6) \Rightarrow (-5 < |x| < 7) \Rightarrow |x| < 7$$

$$-7 < x < 7$$

$$\mathbf{2.}\ \boldsymbol{g})\ \{\boldsymbol{x} \in \mathbb{R}\colon \left|\left||\boldsymbol{x}| - \mathbf{1}\right| - \mathbf{2}\right| - \mathbf{3}\right| < \mathbf{1}\}$$

$$(\boldsymbol{x} > \mathbf{5} \vee \boldsymbol{x} < -\mathbf{5}) \quad \wedge \quad (-\mathbf{7} < \boldsymbol{x} < \mathbf{7})$$

$$\boldsymbol{x} \in (-\mathbf{7}, -\mathbf{5}) \cup (\mathbf{5}, \mathbf{7})$$

$$\mathbf{3.}\,\boldsymbol{a})\ (\boldsymbol{A} \Rightarrow \boldsymbol{B}) \Leftrightarrow (\neg \boldsymbol{B} \Rightarrow \neg \boldsymbol{A})$$

$\boldsymbol{A} \Rightarrow \boldsymbol{B}$

$\boxed{\boldsymbol{A}\text{ je předpoklad}}$ $\boxed{\boldsymbol{B}\text{ je výsledek (závěr)}}$

$\boldsymbol{A}$: **pršet** $\boldsymbol{B}$: **mít deštník**

$\boldsymbol{A} \Rightarrow \boldsymbol{B}$ **Pokud prší, mám deštník**

$\neg \boldsymbol{B} \Rightarrow \neg \boldsymbol{A}$ **Pokud nemám deštník, neprší**

$$3.\ \boldsymbol{a})\ (\boldsymbol{A} \Rightarrow \boldsymbol{B}) \Leftrightarrow (\neg \boldsymbol{B} \Rightarrow \neg \boldsymbol{A})$$

Pokud prší, mám deštník $\Longleftrightarrow$ **Pokud nemám deštník, neprší**

A	B	$A \Rightarrow B$	$\neg B$	$\neg A$	$\neg B \Rightarrow \neg A$
1	1	1	0	0	1
1	0	0	1	0	0
0	1	1	0	1	1
0	0	1	1	1	1

$3. b)\neg(A \Rightarrow B) \Leftrightarrow (A \wedge \neg B)$ **důkazy sporem**

¬(Pokud prší, mám deštník) $\Longleftrightarrow$ **Mám deštník a neprší**

A	B	$A \Rightarrow B$	$\neg(A \Rightarrow B)$	A	$\neg B$	$A \wedge \neg B$
1	1	1	0	1	0	0
1	0	0	1	1	1	1
0	1	1	0	0	0	0
0	0	1	0	0	1	0

$$3.\,e)\ (A \Rightarrow B) \Leftrightarrow (\neg A \vee B)$$

Pokud bude pršet, budu mít deštník $\Longleftrightarrow$ **Nebude pršet nebo budu mít deštník**

A	B	$A \Rightarrow B$	$\neg A$	B	$\neg A \vee B$
1	1	1	0	1	1
1	0	0	0	0	0
0	1	1	1	1	1
0	0	1	1	0	1

$\mathbf{4.}\, \boldsymbol{a})$

$$\forall \boldsymbol{x} \in \boldsymbol{M}\ \exists \boldsymbol{y} \in \boldsymbol{M}: \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y})$$

Každý zná alespoň jednoho člověka

$\boldsymbol{Negace}$:

$$\neg\big(\forall \boldsymbol{x} \in \boldsymbol{M}\ \exists \boldsymbol{y} \in \boldsymbol{M}: \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y})\big)$$

$$\exists \boldsymbol{x} \in \boldsymbol{M}\ \forall \boldsymbol{y} \in \boldsymbol{M}:\ \neg \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y})$$

Existuje člověk, který nikoho nezná

$\mathbf{4.}\ \boldsymbol{b})$

$$\forall \boldsymbol{y} \in \boldsymbol{M}\ \exists \boldsymbol{x} \in \boldsymbol{M}: \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y})$$

Pro každého člověka existuje jeden člověk, který ho zná

Negace:

$$\neg(\forall \boldsymbol{y} \in \boldsymbol{M}\ \exists \boldsymbol{x} \in \boldsymbol{M}: \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y}))$$

$$\exists \boldsymbol{y} \in \boldsymbol{M}\ \forall \boldsymbol{x} \in \boldsymbol{M}: \neg \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y})$$

Existuje jeden člověk, kterého nikdo nezná

$\mathbf{4.}\,\boldsymbol{c})$

$$\forall \boldsymbol{x} \in \boldsymbol{M}\ \forall \boldsymbol{y} \in \boldsymbol{M} : \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y})$$

Každý znají každého

$\boldsymbol{Negace}$:

$$\neg(\forall \boldsymbol{x} \in \boldsymbol{M}\ \forall \boldsymbol{y} \in \boldsymbol{M} : \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y}))$$

$$\exists \boldsymbol{x} \in \boldsymbol{M}\ \exists \boldsymbol{y} \in \boldsymbol{M} : \neg \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y})$$

Existuje alespoň jeden člověk, který nezná alespoň jednoho dalšího člověka

$\mathbf{4}.\,\boldsymbol{d})$

$$\exists \boldsymbol{x} \in \boldsymbol{M}\ \forall \boldsymbol{y} \in \boldsymbol{M}: \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y})$$

Existuje jeden člověk, který zná každého

$\boldsymbol{Negace}$:

$$\neg(\exists \boldsymbol{x} \in \boldsymbol{M}\ \forall \boldsymbol{y} \in \boldsymbol{M}: \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y}))$$

$$\forall \boldsymbol{x} \in \boldsymbol{M}\ \exists \boldsymbol{y} \in \boldsymbol{M}: \neg \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y})$$

Pro každého člověka jako X existuje alespoň jeden další člověk jako Y, takové, že, X nezná Y

$\mathbf{4.}\,\boldsymbol{e})$

$$\exists \boldsymbol{y} \in \boldsymbol{M}\ \forall \boldsymbol{x} \in \boldsymbol{M}: \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y})$$

Exsistuje člověk, kterého každý zná

$\boldsymbol{Negace}$:

$$\neg(\exists \boldsymbol{y} \in \boldsymbol{M}\ \forall \boldsymbol{x} \in \boldsymbol{M}: \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y}))$$

$$\forall \boldsymbol{y} \in \boldsymbol{M}\ \exists \boldsymbol{x} \in \boldsymbol{M}: \neg \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y})$$

Pro každého člověka se najde (existuje) alespoň jeden člověk, který ho nezná

$\mathbf{4.}\ \boldsymbol{f})$

$$\exists \boldsymbol{x} \in \boldsymbol{M}\ \exists \boldsymbol{y} \in \boldsymbol{M}: \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y})$$

Existuje jeden člověk, který zná alespoň jednoho člověka

$\boldsymbol{Negace}$:

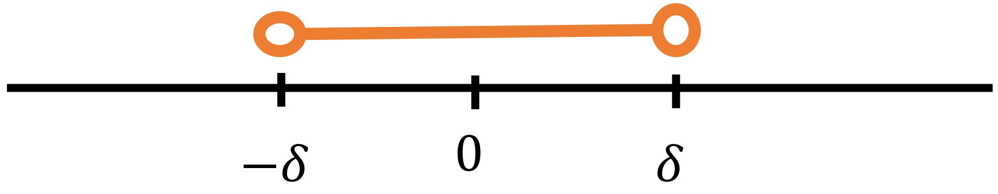
$$\neg(\exists \boldsymbol{x} \in \boldsymbol{M}\ \exists \boldsymbol{y} \in \boldsymbol{M}: \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y})\ )$$

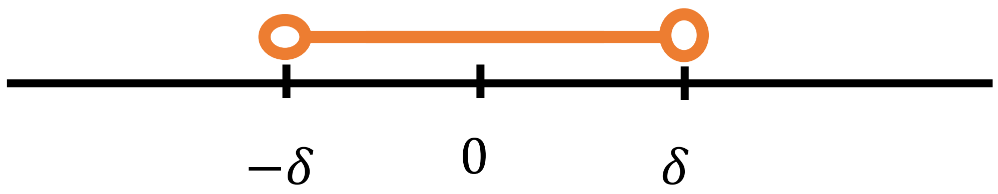
$$\forall \boldsymbol{x} \in \boldsymbol{M}\ \forall \boldsymbol{y} \in \boldsymbol{M}: \neg \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{y})$$

Nikdo nikoho nezná

5. $\boldsymbol{a}$)

$$\forall \boldsymbol{\delta} > \mathbf{0}\ \forall \boldsymbol{x} \in \mathbb{R}: |\boldsymbol{x}| < \boldsymbol{\delta}, \boldsymbol{pr}\text{á}\boldsymbol{v}\text{ě}\ \boldsymbol{kdy}\text{ž}\ (\boldsymbol{x} > -\boldsymbol{\delta}) \wedge (\boldsymbol{x} < \boldsymbol{\delta})$$

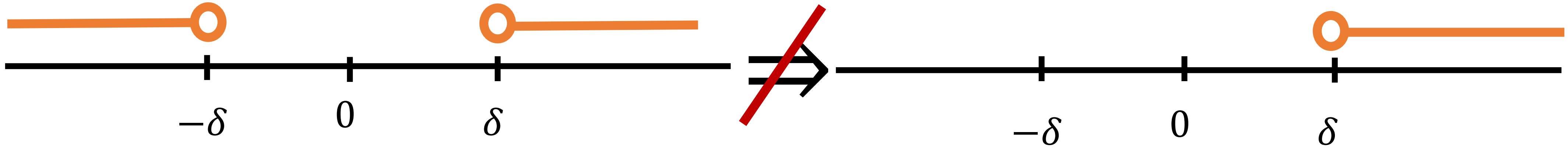




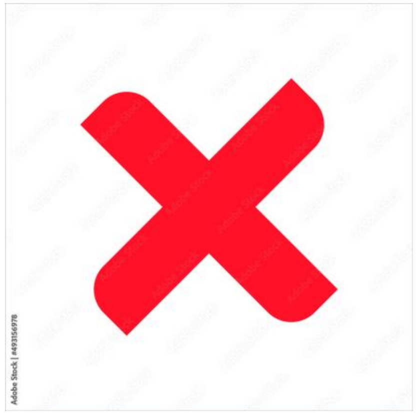
$$\forall \boldsymbol{\delta} > \mathbf{0}\ \forall \boldsymbol{x} \in \mathbb{R}: |\boldsymbol{x}| < \boldsymbol{\delta} \quad \Longleftrightarrow \quad (\boldsymbol{x} > -\boldsymbol{\delta}) \wedge (\boldsymbol{x} < \boldsymbol{\delta})$$

5. *b*)

$$\forall \boldsymbol{\delta} > \mathbf{0}\ \forall \boldsymbol{x} \in \mathbb{R}\colon \textbf{\textit{jestliže}}\ |\boldsymbol{x}| > \boldsymbol{\delta}, \textbf{\textit{pak}}\ \boldsymbol{x} > \boldsymbol{\delta}$$

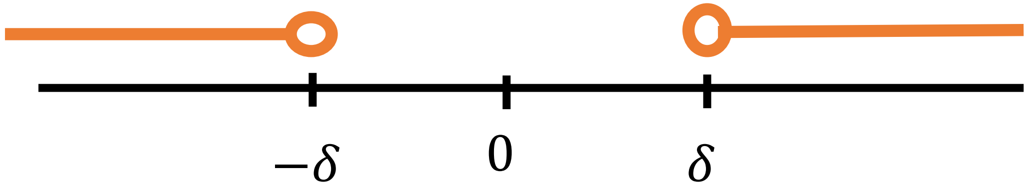


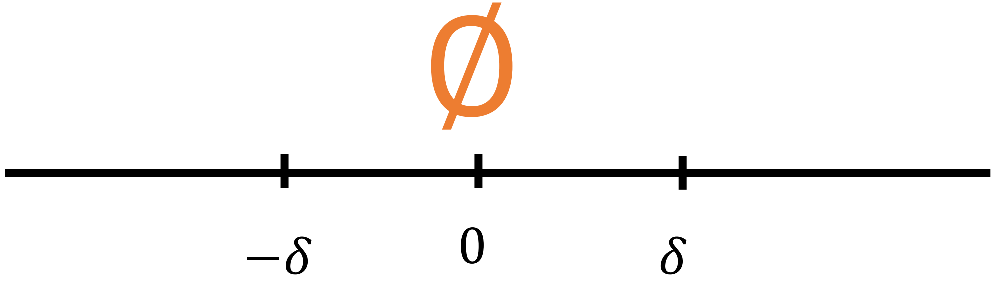
$$|\boldsymbol{x}| > \boldsymbol{\delta} \not\Rightarrow \boldsymbol{x} > \boldsymbol{\delta}$$

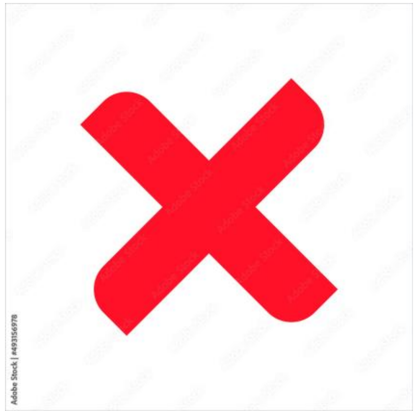


5. c)

$$\forall \boldsymbol{\delta} > \mathbf{0}\ \forall \boldsymbol{x} \in \mathbb{R}\colon |\boldsymbol{x}| > \boldsymbol{\delta},\ \textbf{\textit{právě když}}\ (\boldsymbol{x} < -\boldsymbol{\delta}) \wedge (\boldsymbol{x} > \boldsymbol{\delta})$$

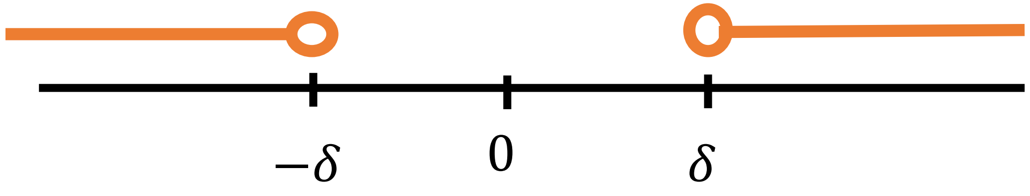


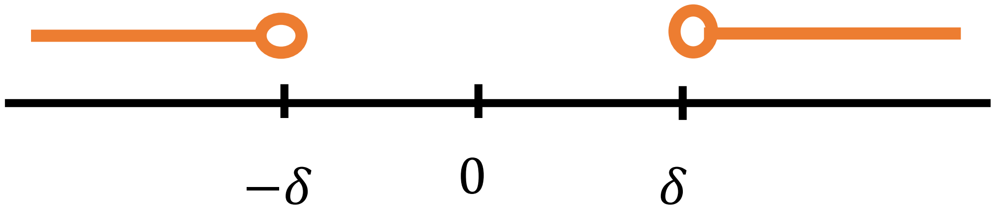




5. *d*)

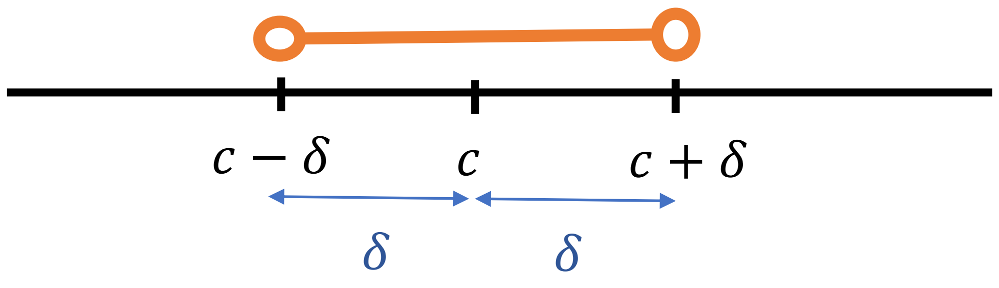
$$\forall \boldsymbol{\delta} > \mathbf{0}\ \forall \boldsymbol{x} \in \mathbb{R}\colon |\boldsymbol{x}| > \boldsymbol{\delta}, \textit{\textbf{právě když}}\ (\boldsymbol{x} < -\boldsymbol{\delta}) \vee (\boldsymbol{x} > \boldsymbol{\delta})$$

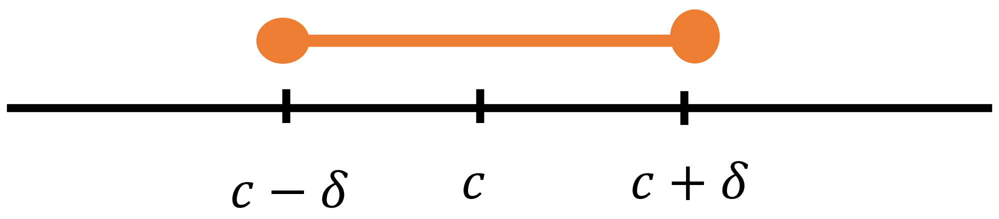


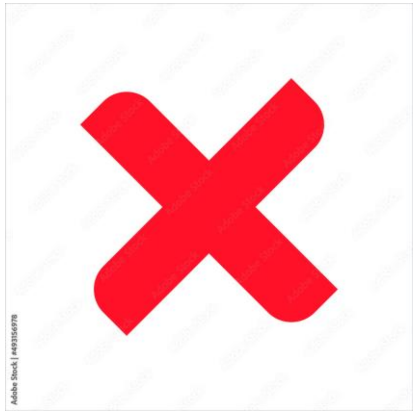


5. $\boldsymbol{e}$)

$$\forall \boldsymbol{\delta} > \mathbf{0}\ \forall \boldsymbol{c} \in \mathbb{R}\ \forall \boldsymbol{x} \in \mathbb{R}: (|\boldsymbol{x}-\boldsymbol{c}| < \boldsymbol{\delta}) \Leftrightarrow (\boldsymbol{x} \in [\boldsymbol{c}-\boldsymbol{\delta}, \boldsymbol{c}+\boldsymbol{\delta}])$$

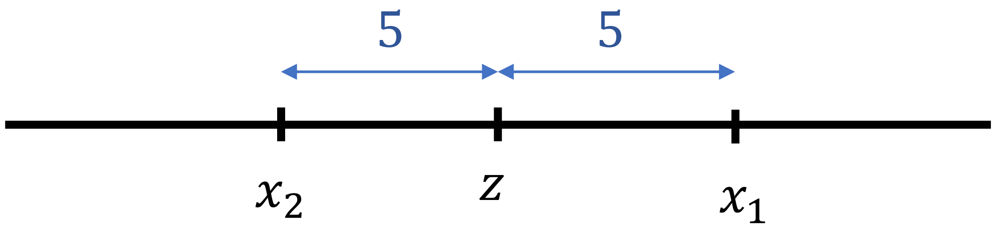






5. *f*)

$$\boldsymbol{\forall z \in \mathbb{R}\ \exists x_1 \in \mathbb{R}\ \exists x_2 \in \mathbb{R}\colon x_1 \neq x_2 \wedge |z - x_1| = 5 \wedge |z - x_1| = 5)}$$



$x_2 = \mathrm{z} - 5$

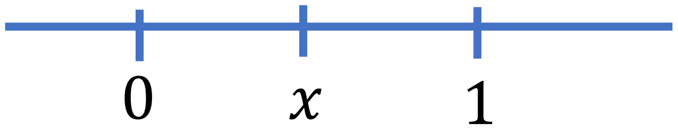
$x_1 = \mathrm{z} + 5$

$$\mathbf{6.}\ \boldsymbol{i})\ \forall x \in \boldsymbol{M}\ \exists y \in \boldsymbol{M}\ \exists z \in \boldsymbol{M}: \boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$$

$\boldsymbol{a})\ \boldsymbol{M} = \mathbb{N} \qquad \boldsymbol{x} = \mathbf{1} \qquad \mathbf{1} = \boldsymbol{y} + \boldsymbol{z}$

$\boldsymbol{b})\ \boldsymbol{M} = \mathbb{N} \cup \{\mathbf{0}\} \qquad \boldsymbol{y} = \boldsymbol{x} \wedge \boldsymbol{z} = \mathbf{0} \qquad \boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$

$\mathbf{c})\ \boldsymbol{M} = (\mathbf{0}, \mathbf{1})$



$$\boldsymbol{y} = \boldsymbol{z} = \frac{\boldsymbol{x}}{\mathbf{2}} : \boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$$

$\boldsymbol{d})\ \boldsymbol{M} = \{\mathbf{0}\} \qquad \boldsymbol{x} = \mathbf{0} \qquad \boldsymbol{y} = \boldsymbol{z} = \mathbf{0} \qquad \boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$

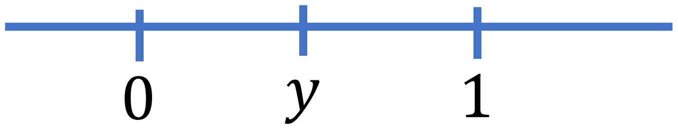
$\boldsymbol{e})\ \boldsymbol{M} = \emptyset$

$6.\, ii)\ \exists y \in M\ \forall x \in M\ \exists z \in M\colon\ x = y + z$

$a)\ M = \mathbb{N} \qquad x = y \Longrightarrow z = 0 \notin \mathbb{N}$

$b)\ M = \mathbb{N} \cup \{0\} \quad y = 0, \qquad z = x \qquad x = y + z$

$c)\ M = (0, 1)$



$x \leq y$

$d)\ M = \{0\} \qquad y = 0 \qquad x = z = 0 \qquad x = y + z$

$e)\ M = \emptyset$

$6.\ iii)\ \exists y \in M\ \exists z \in M\ \forall x \in M:\ x = y + z$

$a)\ M = \mathbb{N}$ $\quad y + z > 1 \quad x = 1$ ✗

$b)\ M = \mathbb{N} \cup \{0\}$ $\quad y + z$ je konstantní ✗

---

c) $M = (0, 1)$ $\quad y + z$ je konstantní ✗

$d)\ M = \{0\}$ $\quad y = z = 0 \quad x = 0 \quad x = y + z$ ✓

$e)\ M = \emptyset$ ✗

$i) \forall x \in M\ \exists y \in M\ \exists z \in M: x = y + z$

6) Shrnutí

$ii)\ \exists y \in M\ \forall x \in M\ \exists z \in M:\ x = y + z$

$iii)\ \exists y \in M\ \exists z \in M\ \forall x \in M:\ x = y + z$

	$a)$	$b)$	$c)$	$d)$	$e)$
$i)$	✗	✓	✓	✓	✓
$ii)$	✗	✓	✗	✓	✗
$iii)$	✗	✗	✗	✓	✗

Výrok iii je nejsilnější

Výrok ii je silnější než i

$\mathbf{7.}\,\boldsymbol{a})$

$\forall \boldsymbol{x} \in \mathbb{R}\ \forall \boldsymbol{y} \in \mathbb{R}\colon\ \boldsymbol{x}^{\mathbf{2}} + \boldsymbol{y}^{\mathbf{2}} > \mathbf{0}$ **Výrok neplatí**

Negace:

$\neg\,(\forall \boldsymbol{x} \in \mathbb{R}\ \forall \boldsymbol{y} \in \mathbb{R}\colon\ \boldsymbol{x}^{\mathbf{2}} + \boldsymbol{y}^{\mathbf{2}} > \mathbf{0})$

$\exists \boldsymbol{x} \in \mathbb{R}\ \exists \boldsymbol{y} \in \mathbb{R}\colon\ \boldsymbol{x}^{\mathbf{2}} + \boldsymbol{y}^{\mathbf{2}} \leq \mathbf{0}$ $\boldsymbol{x} = \boldsymbol{y} = \mathbf{0}\colon\ \boldsymbol{x}^{\mathbf{2}} + \boldsymbol{y}^{\mathbf{2}} = \mathbf{0} \leq \mathbf{0}$

Negace platí

$\mathbf{7.}\,\boldsymbol{b})$

$\forall \boldsymbol{x} \in \mathbb{R}\ \exists \boldsymbol{y} \in \mathbb{N}: (\boldsymbol{y} \leq \boldsymbol{x}) \wedge (\boldsymbol{y} + \mathbf{1} > \boldsymbol{x})$

Výrok neplatí

Negace:

$\neg(\forall \boldsymbol{x} \in \mathbb{R}\ \exists \boldsymbol{y} \in \mathbb{N}: (\boldsymbol{y} \leq \boldsymbol{x}) \wedge (\boldsymbol{y} + \mathbf{1} > \boldsymbol{x}))$

$\exists \boldsymbol{x} \in \mathbb{R}\ \forall \boldsymbol{y} \in \mathbb{N}: (\boldsymbol{y} > \boldsymbol{x}) \vee (\boldsymbol{y} + \mathbf{1} \leq \boldsymbol{x})$

$\boldsymbol{x} = \mathbf{0}: \quad \boldsymbol{y} > \mathbf{0} \quad \vee \quad \boldsymbol{y} \leq -\mathbf{1}$

Negace platí

Důkaz matematickou indukcí

- Důkaz matematickou indukcí se používá pro věty, které se týkají (téměř) všech přirozených čísel nebo jsou na nich nějak závislé.

$$\forall n \in \mathbb{N}: 6 \mid 10^n - 4$$

$$n = 1: \quad 6 \mid (10 - 4) \Rightarrow 6 \mid 6$$

$$n = 2: \quad 6 \mid (100 - 4) \Rightarrow 6 \mid 96$$

Důkaz matematickou indukcí

- Přesněji řečeno, používáme jej pro věty typu $\forall(n \in \mathbb{N}), n \geq n_0: A(n)$, kde:
  - $\boldsymbol{A(n)}$ je výrokový vzorec s volnou proměnnou n reprezentující typicky nějakou vlastnost přirozených čísel
  - $\boldsymbol{n_0}$ je nějaké konkrétní přirozené číslo, od něhož výše má daná vlastnost platit.
- Často má A(n) platit pro každé přirozené číslo, pak je $n_0 = 1$ a podmínku $n \geq n0$ neuvádíme.

Důkaz matematickou indukcí

Důkaz matematickou indukcí sestává ze dvou kroků:

1. Dokážeme, že $A(n)$ platí pro $n = n_0$.
2. Dokážeme implikaci: $\forall(k \in N): A(k) \Rightarrow A(k+1)$, této části říkáme **indukční krok**.

$\mathbf{8.}\,\boldsymbol{c})$

$$\forall \boldsymbol{n} \in \mathbb{N}\colon \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

1. Dokážeme, že rovnice/vzorec platí pro $n = 1$

$$n = 1\colon \sum_{k=1}^{1} k^2 = \frac{1(1+1)(2+1)}{6}$$

$$1^2 = 1$$

indukční předpoklad

$$m \in \mathbb{N}\colon \sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}$$

$$1^2 + 2^2 + \cdots + (m-1)^2 + m^2 = \frac{m(m+1)(2m+1)}{6}$$

$$(m+1) \in \mathbb{N}\colon \sum_{k=1}^{m+1} k^2 = \frac{\overbrace{(m+1)(m+2)(2(m+1)+1)}}{6}$$

$$1^2 + 2^2 + \cdots + (m)^2 + (m+1)^2 = \frac{\overbrace{2m^3 + 9m^2 + 13m + 6}}{6}$$

$$1^2+2^2+\cdots+(m)^2+(m+1)^2=\frac{2m^3+9m^2+13m+6}{6}$$

Podle indukčního předpokladu

$$\overbrace{1^2+2^2+\cdots+(m)^2}+(m+1)^2=\overbrace{\frac{m(m+1)(2m+1)}{6}}+(m+1)^2=$$

$$=\frac{2m^3+3m^2+m}{6}+m^2+2m+1=$$

$$\frac{2m^3+3m^2+m}{6}+\frac{6m^2+12m+6}{6}=\frac{2m^3+9m^2+13m+6}{6}$$

$\mathbf{8.}\ \boldsymbol{d)}$

$\forall \boldsymbol{n} \in \mathbb{N}: \mathbf{6} \mid \mathbf{10}^{\boldsymbol{n}} - \mathbf{4}$

1. Dokážeme, že výrok platí pro $\boldsymbol{n} = \mathbf{1}$

$\boldsymbol{n} = \mathbf{1} : \ \mathbf{6} \mid \mathbf{10}^{\mathbf{1}} - \mathbf{4} \ \Rightarrow \mathbf{6} \mid \mathbf{6}$

indukční předpoklad

$$k \in \mathbb{N}: 6 \mid 10^k - 4 \qquad 10^k - 4 = 6m \qquad 10^k = 6m + 4 \quad m \in \mathbb{N}$$

$$(k+1) \in \mathbb{N}: 6 \mid 10^{k+1} - 4$$

$$10^{k+1} - 4 = 10.\overbrace{10^k}^{} - 4 = 10\,\overbrace{(6m+4)}^{\text{indukční předpoklad}} - 4 = 60m + 40 - 4$$

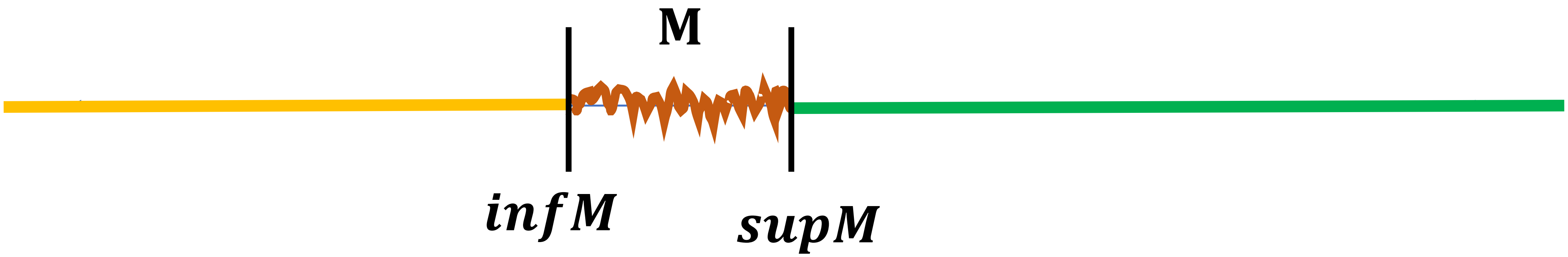
$$= 60m + 36 = 6(10m + 6)$$

$$10^{k+1} - 4 = 6(10m + 6)$$

$$6 \mid 10^{k+1} - 4 \quad ✓$$

Je-li M $\subseteq \mathbb{R}$, pak:

- $\boldsymbol{supM}$ **je nejmenší horní závora**
- $\boldsymbol{infM}$ **je největší dolní závora**



$\boldsymbol{supM = \infty}$, **je-li** $\boldsymbol{M}$ **shora neomezená**

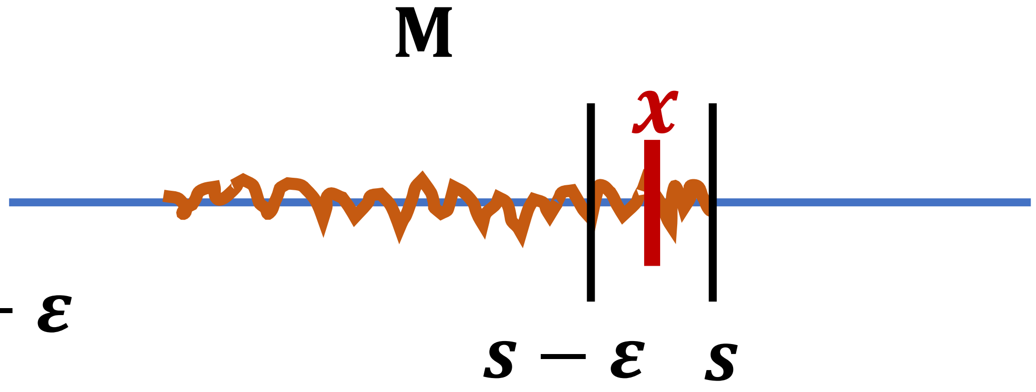
$\boldsymbol{infM = -\infty}$, **je-li** $\boldsymbol{M}$ **zdola neomezená**

$\boldsymbol{sup\emptyset = -\infty}$ $\boldsymbol{inf\emptyset = \infty}$

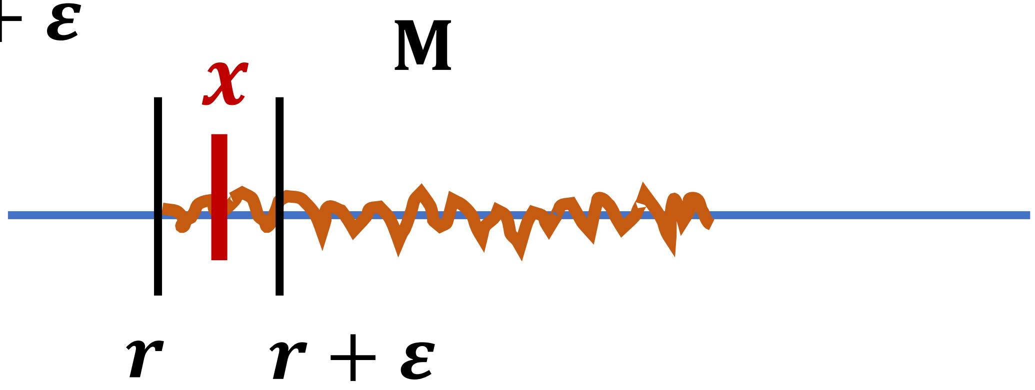
Definice:

Je-li $M \subseteq \mathbb{R}$, $M \neq \emptyset, s, r \in \mathbb{R}$:

$$\text{(i) } \mathbf{s = supM} \iff \begin{cases} s \text{ je horní závora } M \\ \forall \varepsilon > 0 \, \exists x \in M: x > s - \varepsilon \end{cases}$$

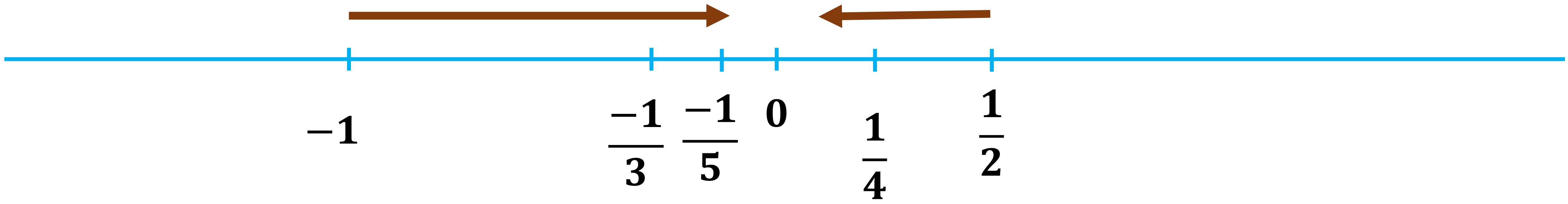


$$\text{(ii) } \mathbf{r = infM} \iff \begin{cases} r \text{ je dolní závora } M \\ \forall \varepsilon > 0 \;\; \exists x \in M: x < r + \varepsilon \end{cases}$$



9) $M = \{\frac{(-1)^n}{n} : n \in \mathbb{N}\}$

$n \in \mathbb{N}$	1	2	3	4	5
$\frac{(-1)^n}{n}$	-1	$\frac{1}{2}$	$\frac{-1}{3}$	$\frac{1}{4}$	$\frac{-1}{5}$



$inf(M) = -1$

$min(M) = -1$

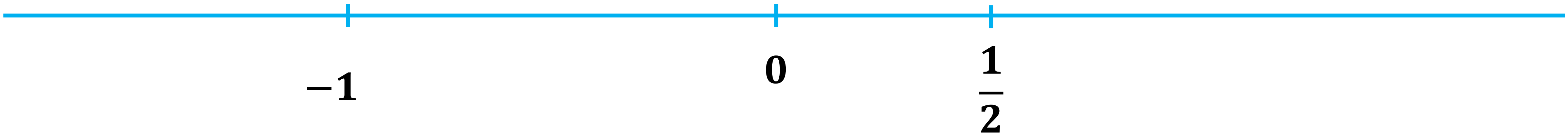
$sup(M) = \frac{1}{2}$

$max(M) = \frac{1}{2}$

9) $M = \{\frac{(-1)^n}{n} : n \in \mathbb{N}\}$

$inf(M) = -1$

$sup(M) = \frac{1}{2}$



$-1 \in M$ **je dolní závora** M $\Rightarrow$**libovolné** $r > -1$ **není dolní závora** M

$\frac{1}{2} \in M$ **je horní závora** M $\Rightarrow$**libovolné** $s < \frac{1}{2}$ **není horní závora** M

$10.\ a)\quad A = [2, \pi)$

$inf(A) = 2$

$min(A) = 2$

$sup(A) = \pi$

$max(A)$

10. b) $B = \mathbb{N}$

$$B = \{1, 2, 3, 4, 5, \ldots\}$$

$$\inf(B) = 1 \qquad \sup(B) = \infty$$

$$\min(B) = 1 \qquad \max(B)$$ ✗

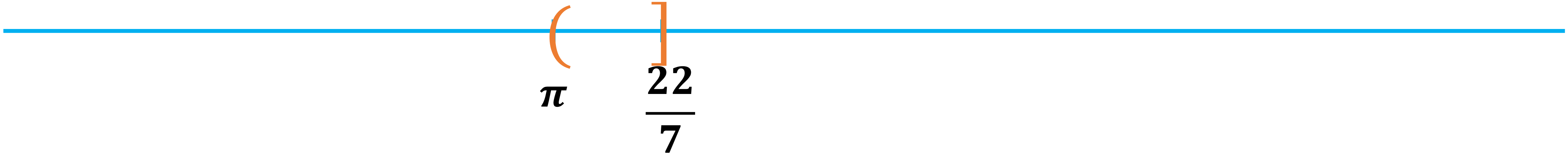
$$\mathbf{10.}\,c)\quad C = \left\{3,\quad 3.1,\quad \sqrt{3},\quad \ln(2),\quad \pi,\quad e,\quad \frac{22}{7}\right\}$$

$$C = \{3,\quad 3.1,\quad 1.732,\quad 0.693,\quad 3.1415,\quad 2.71,\quad 3.1428\}$$

$$inf(C) = \ln(2) \qquad\qquad sup(C) = \frac{22}{7}$$

$$min(C) = \ln(2) \qquad\qquad max(C) = \frac{22}{7}$$

$$10.\ d) \quad D = \left(\pi, \frac{22}{7}\right] \cap \mathbb{Q}$$



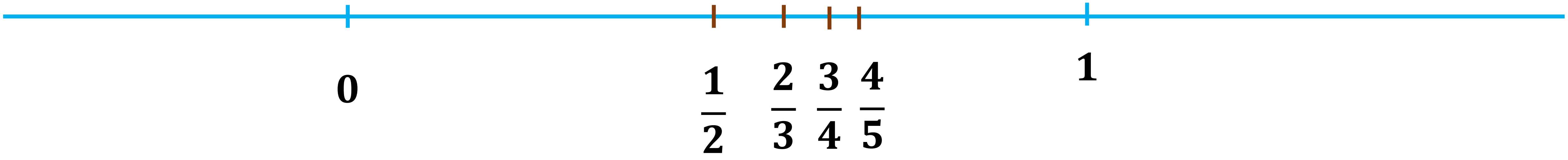
$$\inf(D) = \pi \qquad \sup(D) = \frac{22}{7}$$

$$\min(D) =$$ 

$$\max(D) = \frac{22}{7}$$

$$10.\,e)\quad X = \{\frac{n}{n+1} : n \in \mathbb{N}\}$$

$n \in \mathbb{N}$	1	2	3	4	5	$\longrightarrow$	n
$\frac{n}{n+1}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\longrightarrow$	$\frac{n}{n+1} < 1$



$$inf(X) = \frac{1}{2} \qquad sup(X) = 1$$

$$min(X) = \frac{1}{2} \qquad max(X) =$$

$\mathbf{10.}\,\boldsymbol{e)}\quad \boldsymbol{X = \{\frac{n}{n+1} : n \in \mathbb{N}\}}$

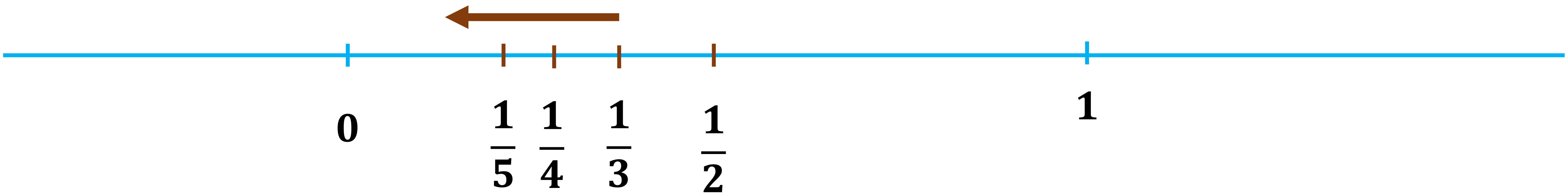
$$\boldsymbol{sup(X) = 1}$$

✓ **1 je horní závora** $\boldsymbol{X}$ $(\forall \boldsymbol{n} \in \mathbb{N}\text{: } \boldsymbol{\frac{n}{n+1} < 1})$

✓ $\forall \boldsymbol{\varepsilon > 0}\ \exists \boldsymbol{n} \in \mathbb{N}\text{: } \boldsymbol{\frac{n}{n+1} > 1 - \varepsilon \iff 1 - \frac{n}{n+1} < \varepsilon \iff \frac{1}{n+1} < \varepsilon \iff n > \frac{1}{\varepsilon} - 1}$

$$10. f) \quad Y = \{1 - \frac{n}{n+1} : n \in \mathbb{N}\} = \{\frac{1}{n+1} : n \in \mathbb{N}\}$$

$n \in \mathbb{N}$	1	2	3	4	5	$\longrightarrow$	n
$\frac{n}{n+1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\longrightarrow$	$\frac{1}{n+1} > 0$



$inf(Y) = \quad 0$ $\qquad$ $sup(Y) = \frac{1}{2}$

$min(Y) =$ $\qquad$ $max(Y) = \frac{1}{2}$

$$\mathbf{10.}\, f) \quad Y = \{1 - \frac{n}{n+1} : n \in \mathbb{N}\} = \{\frac{1}{n+1} : n \in \mathbb{N}\}$$

$$inf(Y) = 0$$

- ✓ **0 je dolní závora** $(\forall n \in \mathbb{N}: \frac{1}{n+1} > 0)$

- ✓ $\forall \varepsilon > 0\ \exists n \in \mathbb{N}: \frac{1}{n+1} < 0 + \varepsilon \iff n + 1 > \frac{1}{\varepsilon} \iff n > \frac{1}{\varepsilon} - 1$

$$10.\,g)\quad Z=\{x<\frac{1}{x}:x\in\mathbb{R}\}$$

$$x^2-1=0\ \Rightarrow x=1\ \vee x=-1$$

$$x<\frac{1}{x}\ \Leftrightarrow x-\frac{1}{x}<0\ \ \Leftrightarrow \frac{x^2-1}{x}<0$$

$$x=0$$

	$(-\infty,-1)$	$[-1,0)$	$[0,1)$	$[1,\infty)$
x^2-1	$+$	0 $-$	$-$	0 $+$
x	$-$	$-$	0 $+$	$+$
$\frac{x^2-1}{x}$	$-$ ✓	0 $+$	∞ $-$ ✓	0 $+$

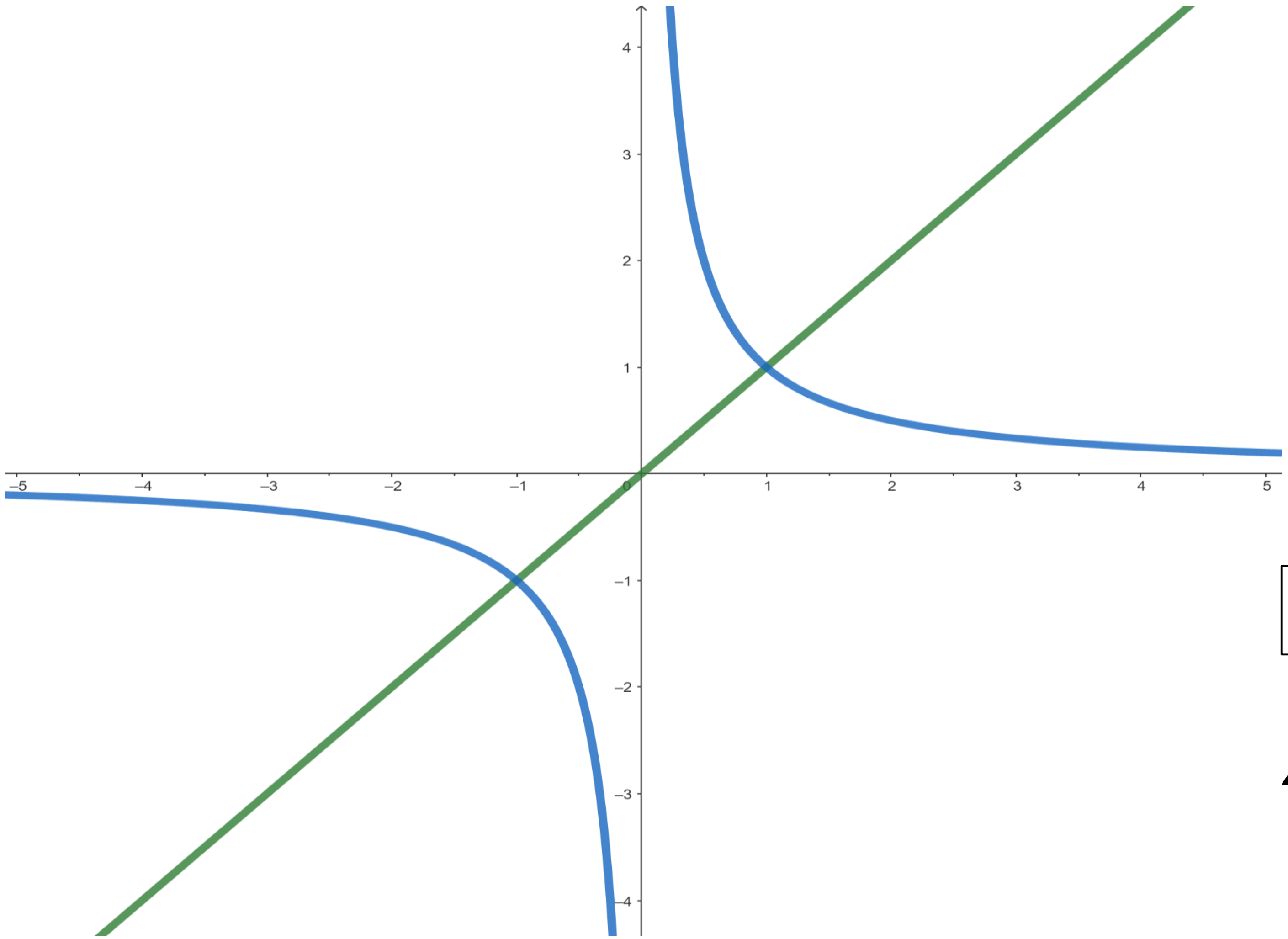
$$Z=(-\infty,-1)\cup(0,1)$$

$$\boxed{x\in(-\infty,0)\cup(0,1)}$$

$$\mathbf{10.}\,g)\quad Z = \left\{x < \frac{1}{x} : x \in \mathbb{R}\right\}$$

$f(x) = x$

$g(x) = \dfrac{1}{x}$



$$\boxed{x \in (-\infty, -1) \cup (0, 1)}$$

$$Z = (-\infty, -1) \cup (0, 1)$$

$$\mathbf{10.}\,g) \quad Z = \left\{x < \frac{1}{x} : x \in \mathbb{R}\right\}$$

$$Z = (-\infty, -1) \cup (0, 1)$$

$\inf(Z) = -\infty$

$\sup(Z) = 1$

$\min(Z)$

$\max(Z)$

$$10.\ h)\quad W = \{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\}$$

$$\frac{1}{n} - \frac{1}{m}$$

$$n = 1:\ \frac{1}{n} - \frac{1}{m} = 1 - \frac{1}{m}$$

$$sup(W) = 1$$

$$\frac{1}{n} - \frac{1}{m}$$

$$m = 1:\ \frac{1}{n} - \frac{1}{m} = \frac{1}{n} - 1$$

$$inf(W) = -1$$

$min(W)$ ❌

$max(W)$ ❌

11)

$$\boldsymbol{sup\Omega \geq inf\Omega}$$

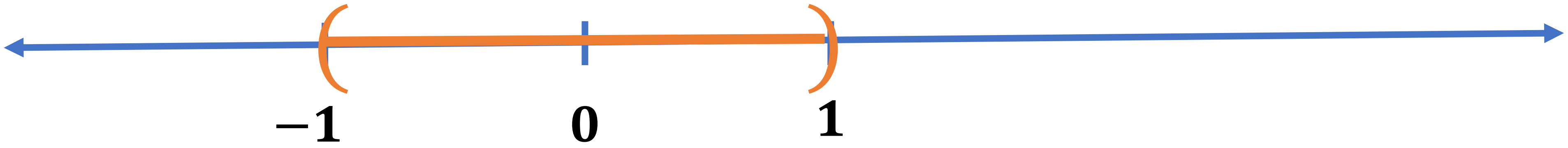
$$\boldsymbol{\Omega} = \emptyset$$

$$\boldsymbol{sup}\emptyset = -\infty$$

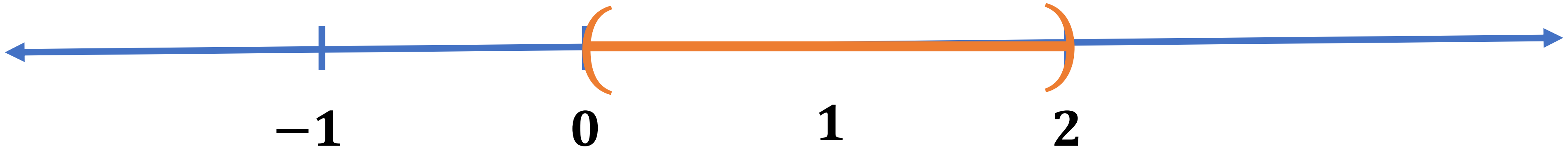
$$\boldsymbol{inf}\emptyset = \infty$$

$12.\ a)\ A \cup B \quad inf(A) = -1 \quad sup(A) = 1$

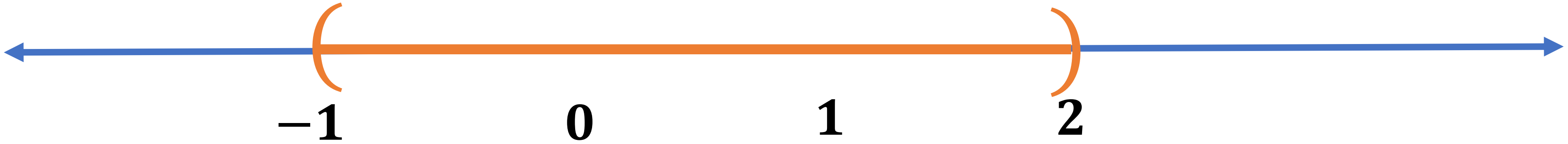
$A = (-1, 1)$



$inf(B) = 0 \quad B = (0, 2) \quad sup(B) = 2$



$inf(A \cup B) = -1 \quad (A \cup B) = (-1, 2) \quad sup(A \cup B) = 2$



$inf(A \cup B) = min\{inf(A), inf(B)\} \quad sup(A \cup B) = max\{sup(A), sup(B)\}$

$12. a)\ A \cup B$ $\boxed{sup(A \cup B) = max\{sup(A), sup(B)\}}$

- $s := max\{supA, supB\}$, s je horní závora $A \cup B$. Proč?

$$\forall x \in A \cup B \Longrightarrow \begin{cases} x \in A \ \Rightarrow x \leq supA \\ x \in B \ \Rightarrow x \leq supB \end{cases} \Rightarrow x \leq s$$

- $\forall \varepsilon > 0 \ \exists x \in A \cup B\colon\ x > s - \varepsilon$

$$\varepsilon > 0 \ \begin{cases} s = sup\mathrm{A} \ \Rightarrow \exists x \in A{:}\, x > supA - \varepsilon = s - \varepsilon \\ s = sup\mathrm{B} \ \Rightarrow \exists x \in B{:}\, x > supB - \varepsilon = s - \varepsilon \end{cases} \Rightarrow \exists x \in A \cup B\colon\ x > s - \varepsilon$$

$12.\ a)\ A \cup B$ $\boxed{inf(A \cup B) = max\{inf A, inf B\}}$

- $r := min\{inf A, inf B\}$, r je dolní závora $A \cup B$. Proč?

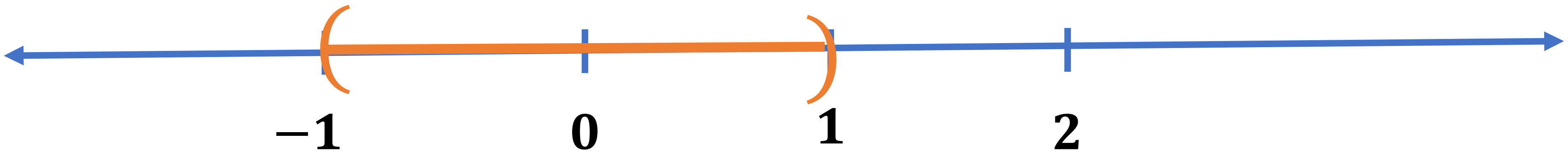
$$\forall x \in A \cup B \Longrightarrow \begin{cases} x \in A \Rightarrow x \geq inf A \\ x \in B \Rightarrow x \geq inf B \end{cases} \Rightarrow x \geq r$$

- $\forall \varepsilon > 0\ \exists x \in A \cup B\!: \ x < r + \varepsilon$

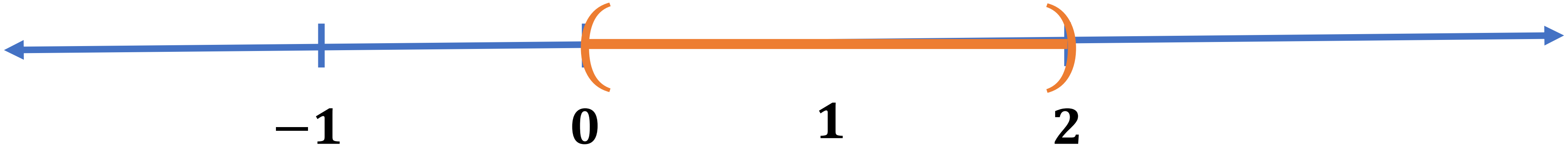
$$\varepsilon > 0 \quad \begin{cases} r = inf \mathrm{A} \Rightarrow \exists x \in A\!: x < inf A + \varepsilon = r + \varepsilon \\ r = inf \mathrm{B} \Rightarrow \exists x \in B\!: x < inf B + \varepsilon = r + \varepsilon \end{cases} \Rightarrow \exists x \in A \cup B\!: \ x < r + \varepsilon$$

$12.b)\ A \cap B$

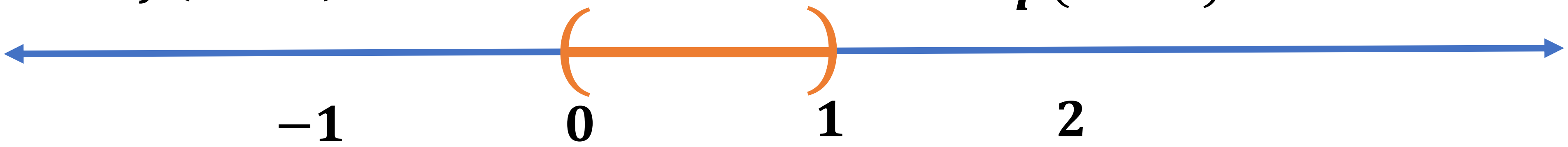
$inf(A) = -1$ $\quad A = (-1, 1)$ $\quad sup(A) = 1$



$inf(B) = 0$ $\quad B = (0, 2)$ $\quad sup(B) = 2$



$inf(A \cap B) = 0$ $\quad (A \cap B) = (0, 1)$ $\quad sup(A \cap B) = 1$



$$inf(A \cap B) = max\{inf(A), inf(B)\} \qquad sup(A \cap B) = min\{sup(A), sup(B)\}$$

$$12.\, b)\ A \cap B$$

$$A = \{-1, 0, 1\} \qquad inf(A) = -1 \qquad sup(A) = 1$$

$$B = \{-2, 0, 2\} \qquad inf(B) = -2 \qquad sup(B) = 2$$

$$A \cap B = \{0\} \qquad inf(A \cap B) = 0 \qquad sup(A \cap B) = 0$$

$$inf(A \cap B) > max\{inf(A), inf(B)\} \qquad sup(A \cap B) < min\{sup(A), sup(B)\}$$

$\mathbf{12}.\,\boldsymbol{b})\ \boldsymbol{A \cap B}$

$$\boldsymbol{inf(A \cap B) \geq max\{inf(A), inf(B)\}}$$

$$\boldsymbol{sup(A \cap B) \leq min\{sup(A), sup(B)\}}$$

$12.b)\ A \cap B$

$$\boxed{sup(A \cap B) \leq min\{sup(A), sup(B)\}}$$

$\forall x \in (A \cap B): x \in A \Rightarrow x \leq sup(A)$ $\Leftrightarrow$ $sup(A)$ **je horní zavora** $A \cap B$

$\forall x \in (A \cap B): x \in B \Rightarrow x \leq sup(B)$ $\Leftrightarrow$ $sup(B)$ **je horní zavora** $A \cap B$

$sup(A \cap B)$ **je nejmenší horní závora** $A \cap B$

$$sup(A \cap B) \leq min\{sup(A), sup(B)\}$$

$\mathbf{12.}\,b)\ A \cap B$

$$\boxed{inf(A \cap B) \geq max\{inf(A), inf(B)\}}$$

$\forall x \in (A \cap B)\colon\ x \in A \Rightarrow x \geq inf(A)$ $\Leftrightarrow$ $inf(A)$ **je dolní závora** $A \cap B$

$\forall x \in (A \cap B)\colon\ x \in B \Rightarrow x \geq inf(B)$ $\Leftrightarrow$ $inf(B)$ **je horní závora** $A \cap B$

$inf(A \cap B)$ **je největší horní závora** $A \cap B$

$$inf(A \cap B) \geq max\{inf(A), inf(B)\}$$

$12.c)$ $A+B$ $\quad A+B=\{x+y \mid x\in A \wedge y\in B\}$

$\forall x\in A\ y\in B$: $\quad x+y\geq \inf A+\inf B \quad \Rightarrow \inf(A)+\inf(B)$ je dolní závora $A+B$

$\inf(A+B)$ je největší dolní závora $A+B$ $\boxed{\Rightarrow \inf(A)+\inf(B)\leq \inf(A+B)}$

$\forall x\in A\ y\in B: \inf(A+B)\leq x+y \quad \Leftrightarrow \forall x\in A\ y\in B: \underbrace{\inf(A+B)-x}\leq y$

$\boxed{\inf(A+B)-x \text{ je dolní závora } B}$

$\Leftrightarrow \forall x\in A: \inf(A+B)-x\leq \inf(B) \ \Leftrightarrow \forall x\in A: \underbrace{\inf(A+B)-\inf(B)}\leq x$

$\boxed{\inf(A+B)-\inf(B) \text{ je dolní závora A}}$

$\Leftrightarrow \inf(A+B)-\inf(B)\leq \inf(A)$ $\boxed{\Leftrightarrow \inf(A+B)\leq \inf(A)+\inf(B)}$

$$\boxed{\inf(A+B)=\inf(A)+\inf(B)}$$

$12.c)\ A+B \qquad A+B=\{a+b \mid a\in A \wedge b\in B\}$

$\forall x\in A \wedge y\in B:\ x+y\le supA+supB \Rightarrow supA+supB$ je horní závora $A+B$

$sup(A+B)$ je nejmenší horní závora $A+B$ $\boxed{\Rightarrow sup(A+B)\le supA+supB}$

$\forall x\in A\ y\in B: x+y\le sup(A+B) \quad \Leftrightarrow \forall x\in A\ y\in B: \underbrace{sup(A+B)-x}\ge y$

$sup(A+B)-x$ je horní závora B

$\Leftrightarrow \forall x\in A: supB\le sup(A+B)-x \quad \Leftrightarrow \forall x\in A: \underbrace{sup(A+B)-supB}\ge x$

$sup(A+B)-sup(B)$ je horní závora A

$\Leftrightarrow sup(A+B)-supB\ge supA$ $\boxed{\Leftrightarrow sup(A+B)\ge supA+supB}$

$$\boxed{sup(A+B)=supA+supB}$$

$\mathbf{12.}\ d)\ A \setminus B \qquad A\setminus B = \{x: x \in A \wedge x \notin B\}$

$$\forall x \in A\setminus \mathrm{B} \Rightarrow x \in A \quad \Rightarrow x \leq supA \quad \left.\begin{array}{l} \Rightarrow supA \text{ je horní závora } A\setminus B \\ sup(A\setminus B) \text{ je nejmenší horní závora } A\setminus B \end{array}\right\} \Rightarrow$$

$$\Rightarrow sup(A\setminus B) \leq supA$$

$\mathbf{12.}\,\boldsymbol{d})\ \boldsymbol{A \setminus B}$ $\qquad \boldsymbol{A\setminus B = \{x\colon x \in A \ \wedge x \notin B\}}$

$\forall \boldsymbol{x} \in \boldsymbol{A}\setminus\mathrm{B} \Rightarrow\ \boldsymbol{x} \in \boldsymbol{A} \ \ \Rightarrow \boldsymbol{x} \geq \boldsymbol{infA}$ $\quad \Rightarrow \boldsymbol{infA}$ **je dolní závora** $\boldsymbol{A\setminus B}$

$\boldsymbol{inf(A\setminus B)}$ **je největší dolní závora** $\boldsymbol{A\setminus B}$ $\quad \Rightarrow$

$\Rightarrow \boldsymbol{inf(A\setminus B) \geq infA}$

13. (i)

Číslo $\boldsymbol{h} \in \mathbb{R}$ je horní závora $\boldsymbol{A} \Leftrightarrow \forall \boldsymbol{x} \in \boldsymbol{A}: \boldsymbol{x} \leq \boldsymbol{h} \ \Leftrightarrow \forall(-\boldsymbol{x}) \in -\boldsymbol{A}: -\boldsymbol{x} \geq -\boldsymbol{h}$

$\Leftrightarrow -\boldsymbol{h} \in \mathbb{R}$ je dolní závora $\boldsymbol{A}$

13. (ii)

$\boldsymbol{A}$ je shora omezená $\Leftrightarrow \exists \boldsymbol{K} \in \mathbb{R}\, \forall \boldsymbol{x} \in \boldsymbol{A}: \boldsymbol{x} \leq \boldsymbol{K} \ \Leftrightarrow \exists(-\boldsymbol{K}) \in \mathbb{R}\, \forall -\boldsymbol{x} \in -\boldsymbol{A}: -\boldsymbol{x} \geq -\boldsymbol{K}$

$\Leftrightarrow -\boldsymbol{A}$ je zdola omezená

$\mathbf{13.}\ (\boldsymbol{iii})$

$$\boldsymbol{inf(-A) = r} \iff \begin{cases} \boldsymbol{r} \text{ je dolní závora } \boldsymbol{-A} \\ \forall \boldsymbol{\varepsilon} > \mathbf{0}\ \exists \boldsymbol{x} \in \boldsymbol{-A}: \boldsymbol{x} < \boldsymbol{r} + \boldsymbol{\varepsilon} \end{cases} \iff \begin{cases} \boldsymbol{-r} \text{ je horní závora } \boldsymbol{A} \\ \forall \boldsymbol{\varepsilon} > \mathbf{0}\ \exists - \boldsymbol{x} \in \boldsymbol{A}: \boldsymbol{-x} > \boldsymbol{-r} - \boldsymbol{\varepsilon} \end{cases}$$

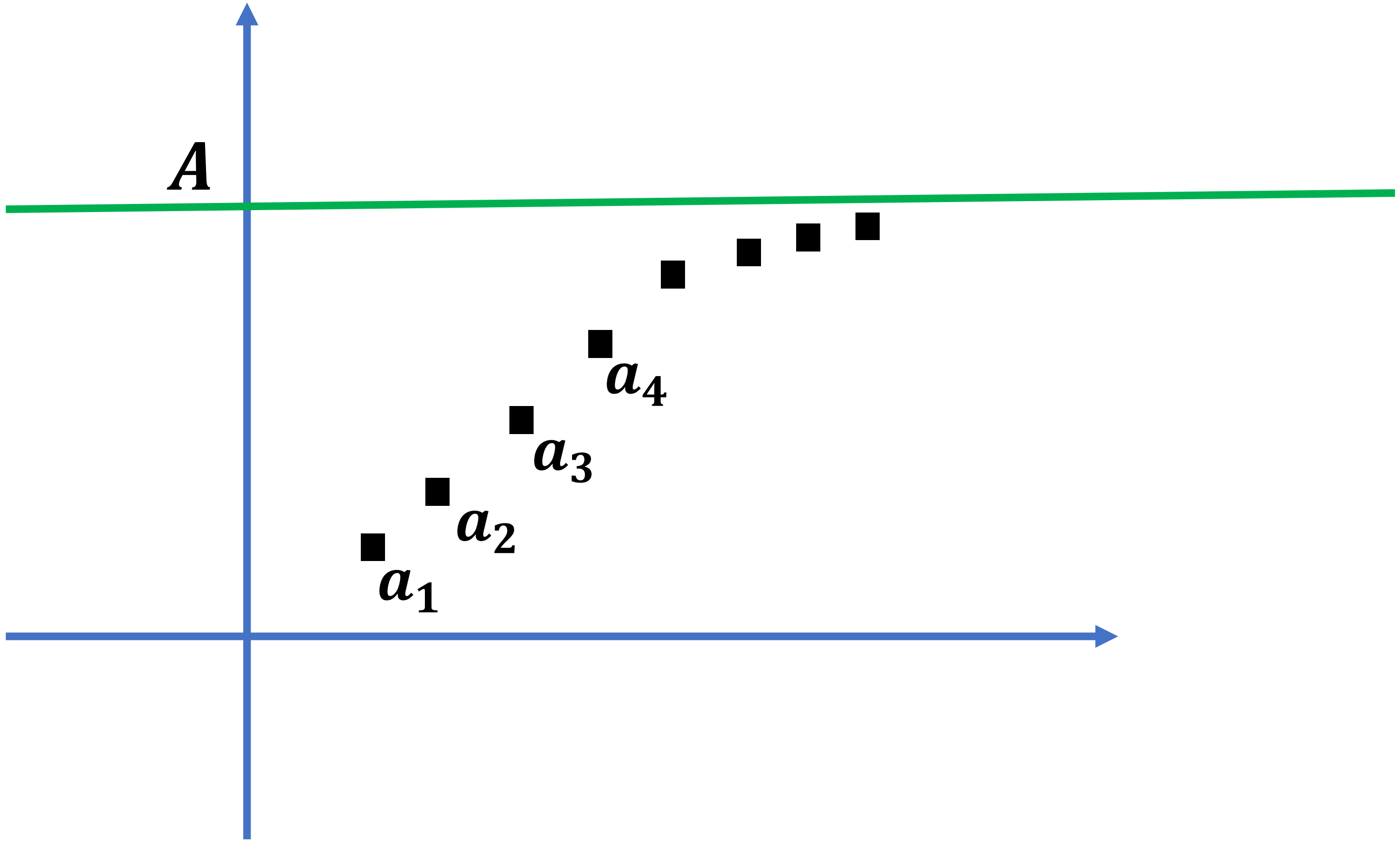
$$\iff \boldsymbol{-r} \text{ je supremum množiny } \boldsymbol{A} \iff \boldsymbol{-r = sup(A)} \iff \boldsymbol{r = -sup(A)}$$

$$\left.\begin{matrix} \boldsymbol{r = inf(-A)} \\ \boldsymbol{r = -sup(A)} \end{matrix}\right. \iff \mathbf{inf(-A) = -sup(A)}$$

LIMITA POSLOUPNOSTI

$$\{a_n\}_{n=1}^{\infty}$$

$$\lim_{n \to \infty} a_n = A$$



$$\lim_{n\to\infty}\frac{1}{n} = 0 \qquad \lim_{n\to\infty} n = \infty$$

$$2\times\infty = \infty \qquad (-3)\times\infty = -\infty \qquad \infty - 6 = \infty \qquad \infty\times\infty = \infty$$

Nedefinujeme: $\infty\times 0 \qquad \frac{\infty}{\infty} \qquad \infty^0 \qquad 0^0 \qquad \frac{0}{0}$

Věta o aritmetice limit

(VOAL)

Jsou-li $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ **posloupnost,**

takové, že $\lim\limits_{n\to\infty} a_n = A, \lim\limits_{n\to\infty} b_n = B.$ **Pak:**

(i) $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = A \pm B$

(ii) $\lim\limits_{n\to\infty}(a_n . b_n) = \lim\limits_{n\to\infty} a_n . \lim\limits_{n\to\infty} b_n = A . B$

(iii) $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \dfrac{A}{B}$ **(pokud** $B \neq 0$**)**

$$\lim_{n \to \infty} c = c$$

$$\{5\}_{n=1}^{\infty}$$

$$5, 5, 5, 5, 5, \ldots \qquad \lim_{n \to \infty} 5 = 5$$

Vypočtěte $\lim\limits_{n\to\infty} \frac{3n+5}{7n+11} = \frac{\infty}{\infty}$

$$\lim_{n\to\infty} \frac{3n+5}{7n+11} \neq \frac{\lim\limits_{n\to\infty} 3n+5}{\lim\limits_{n\to\infty} 7n+11}$$

$$\lim_{n\to\infty} \frac{3n+5}{7n+11} = \lim_{n\to\infty} \frac{\cancel{n}(3+\frac{5}{n})}{\cancel{n}(7+\frac{11}{n})} = \lim_{n\to\infty} \frac{3+\frac{5}{n}}{7+\frac{11}{n}} \stackrel{\cancel{?}}{=} \frac{\lim\limits_{n\to\infty}(3+\frac{5}{n})}{}$$

$$\stackrel{\cancel{?}}{=} \frac{\lim\limits_{n\to\infty} 3 + \lim\limits_{n\to\infty} \frac{5}{n}}{\lim\limits_{n\to\infty} 7 + \lim\limits_{n\to\infty} \frac{11}{n}} \stackrel{\cancel{?}}{=} \frac{3 + \lim\limits_{n\to\infty} 5 \,.\, \lim\limits_{n\to\infty} \frac{1}{n}}{7 + \lim\limits_{n\to\infty} 11 \,.\, \lim\limits_{n\to\infty} \frac{1}{n}} = \frac{3+(5)(0)}{7+(11)(0)} = \frac{3}{7}$$

Vypočtěte $\lim\limits_{n\to\infty}\frac{n^2}{n^2+1} = \frac{\infty}{\infty}$

$$\lim_{n\to\infty}\frac{n^2}{n^2+1} \neq \frac{\lim\limits_{n\to\infty} n^2}{\lim\limits_{n\to\infty} n^2+1}$$

$$\lim_{n\to\infty}\frac{n^2}{n^2+1} = \lim_{n\to\infty}\frac{\cancel{n^2}}{\cancel{n^2}(1+\frac{1}{n^2})} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}} = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty}(1+\frac{1}{n^2})}$$

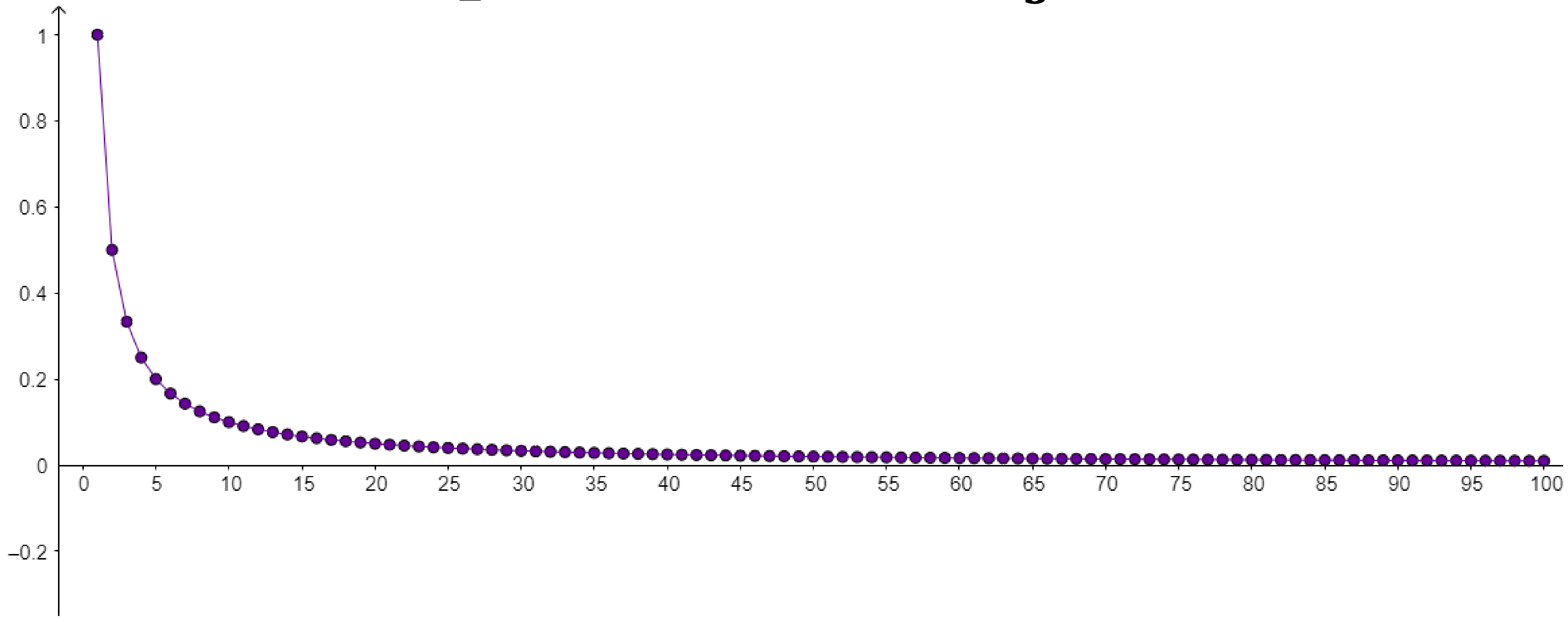
$$= \frac{1}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty}\frac{1}{n^2}} = \frac{1}{1+\lim\limits_{n\to\infty}(\frac{1}{n}\cdot\frac{1}{n})} = \frac{1}{1+\lim\limits_{n\to\infty}(\frac{1}{n})\cdot\lim\limits_{n\to\infty}(\frac{1}{n})} = \frac{1}{1+0} = 1$$

Definice (Vlastní limita posloupnosti)**.** Nechť $\{a_n\}_{n=1}^{\infty}$ je posloupnost. Řekneme, že číslo A je limita posloupnosti $\{a_n\}_{n=1}^{\infty}$ a píšeme $\lim_{n\to\infty} a_n = A$ jestliže platí:

$$\exists n_0 \in \mathbb{N} \ \forall n \geqslant n_0 : |a_n - A| < \varepsilon$$

$$1.\ a)\ \lim_{n\to\infty} \frac{1}{n} = 0$$

$n \in \mathbb{N}$	1	2	3	4	5	$\longrightarrow$	∞
$\frac{1}{n}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\longrightarrow$	0



$$\mathbf{1.}\,\boldsymbol{a})\ \lim_{n\to\infty}\frac{\mathbf{1}}{\boldsymbol{n}}=\mathbf{0}$$

$$\forall\boldsymbol{\varepsilon}>\mathbf{0}\ \exists\boldsymbol{n_0}\in\mathbb{N}\ \forall\boldsymbol{n}\geq\boldsymbol{n_0}\colon\ \left|\frac{\mathbf{1}}{\boldsymbol{n}}-\mathbf{0}\right|<\boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon}>\mathbf{0}\colon\qquad\left|\frac{\mathbf{1}}{\boldsymbol{n}}-\mathbf{0}\right|<\boldsymbol{\varepsilon}\quad\Leftrightarrow\frac{\mathbf{1}}{\boldsymbol{n}}<\boldsymbol{\varepsilon}\quad\Leftrightarrow\boldsymbol{n}>\frac{\mathbf{1}}{\boldsymbol{\varepsilon}}$$

$$\boldsymbol{\varepsilon}>\mathbf{0}\colon\qquad\boldsymbol{n_0}\in\mathbb{N}\wedge\boldsymbol{n_0}>\frac{\mathbf{1}}{\boldsymbol{\varepsilon}}$$

$$\boldsymbol{\varepsilon}=\mathbf{0.12}\Rightarrow\qquad\frac{\mathbf{1}}{\boldsymbol{\varepsilon}}\approx\mathbf{8.33}\qquad\boldsymbol{n_0}\geq\mathbf{9}$$

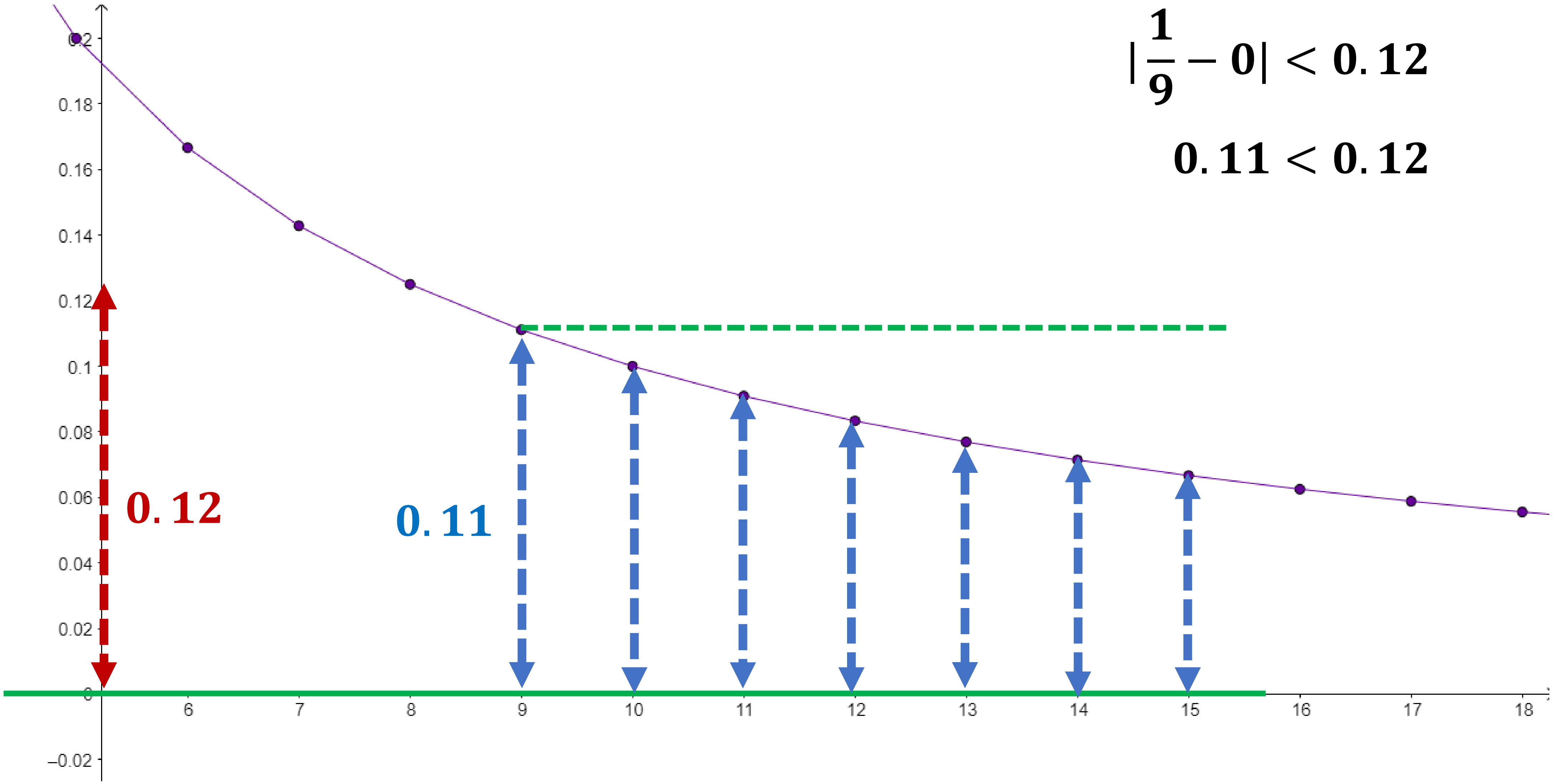
$$\boldsymbol{\varepsilon}=\mathbf{0.011}\Rightarrow\qquad\frac{\mathbf{1}}{\boldsymbol{\varepsilon}}\approx\mathbf{90.9}\qquad\boldsymbol{n_0}\geq\mathbf{91}$$

$$\mathbf{1.}\,a)\ \lim_{n\to\infty}\frac{1}{n}=0 \qquad \varepsilon = 0.12 \qquad n_0 = 9$$

$$\left|\frac{1}{n_0}-0\right| < 0.12$$

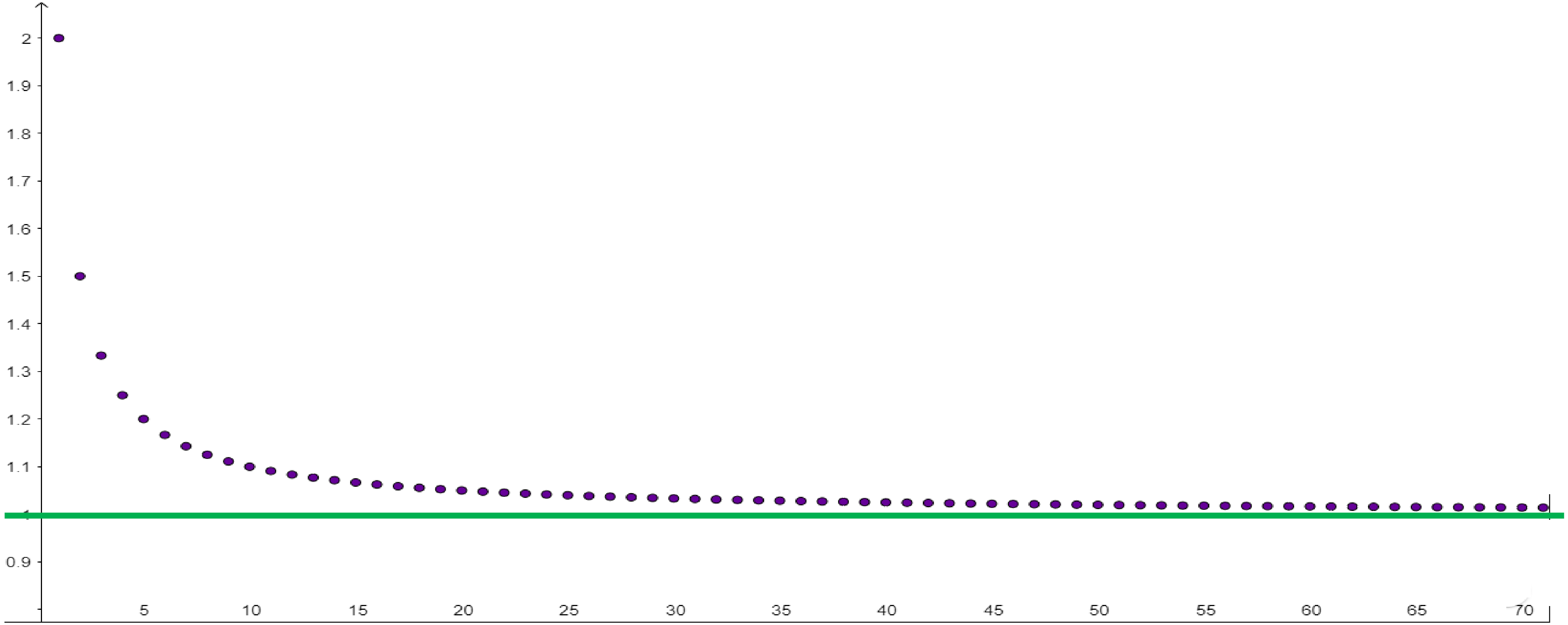
$$\left|\frac{1}{9}-0\right| < 0.12$$

$$0.11 < 0.12$$



$$1.\, b)\ \lim_{n\to\infty} \frac{n+1}{n} = 1$$

$n \in \mathbb{N}$	1	2	3	4	5	$\longrightarrow$	∞
$\frac{1}{n}$	2	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{6}{5}$	$\longrightarrow$	1



$$\mathbf{1}.\,\boldsymbol{b})\ \lim_{\boldsymbol{n}\to\infty}\frac{\boldsymbol{n}+\mathbf{1}}{\boldsymbol{n}}=\mathbf{1}$$

$$\forall\boldsymbol{\varepsilon}>\mathbf{0}\ \ \exists\boldsymbol{n}_{\mathbf{0}}\in\mathbb{N}\ \ \forall\boldsymbol{n}\geq\boldsymbol{n}_{\mathbf{0}}:\ \ \left|\frac{\boldsymbol{n}+\mathbf{1}}{\boldsymbol{n}}-\mathbf{1}\right|<\boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon}>\mathbf{0}:\qquad\left|\frac{\boldsymbol{n}+\mathbf{1}}{\boldsymbol{n}}-\mathbf{1}\right|<\boldsymbol{\varepsilon}\Leftrightarrow\left|\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{n}}-\mathbf{1}\right|<\boldsymbol{\varepsilon}\ \Leftrightarrow\frac{\mathbf{1}}{\boldsymbol{n}}<\boldsymbol{\varepsilon}\ \Leftrightarrow\boldsymbol{n}>\frac{\mathbf{1}}{\boldsymbol{\varepsilon}}$$

$$\boldsymbol{\varepsilon}>\mathbf{0}:\qquad\boldsymbol{n}_{\mathbf{0}}\in\mathbb{N}\vee\boldsymbol{n}_{\mathbf{0}}>\frac{\mathbf{1}}{\boldsymbol{\varepsilon}}$$

$$\boldsymbol{\varepsilon}=\mathbf{0.14}\Rightarrow\frac{\mathbf{1}}{\boldsymbol{\varepsilon}}\approx\mathbf{7.14}\qquad\qquad\boldsymbol{n}_{\mathbf{0}}\geq\mathbf{8}$$

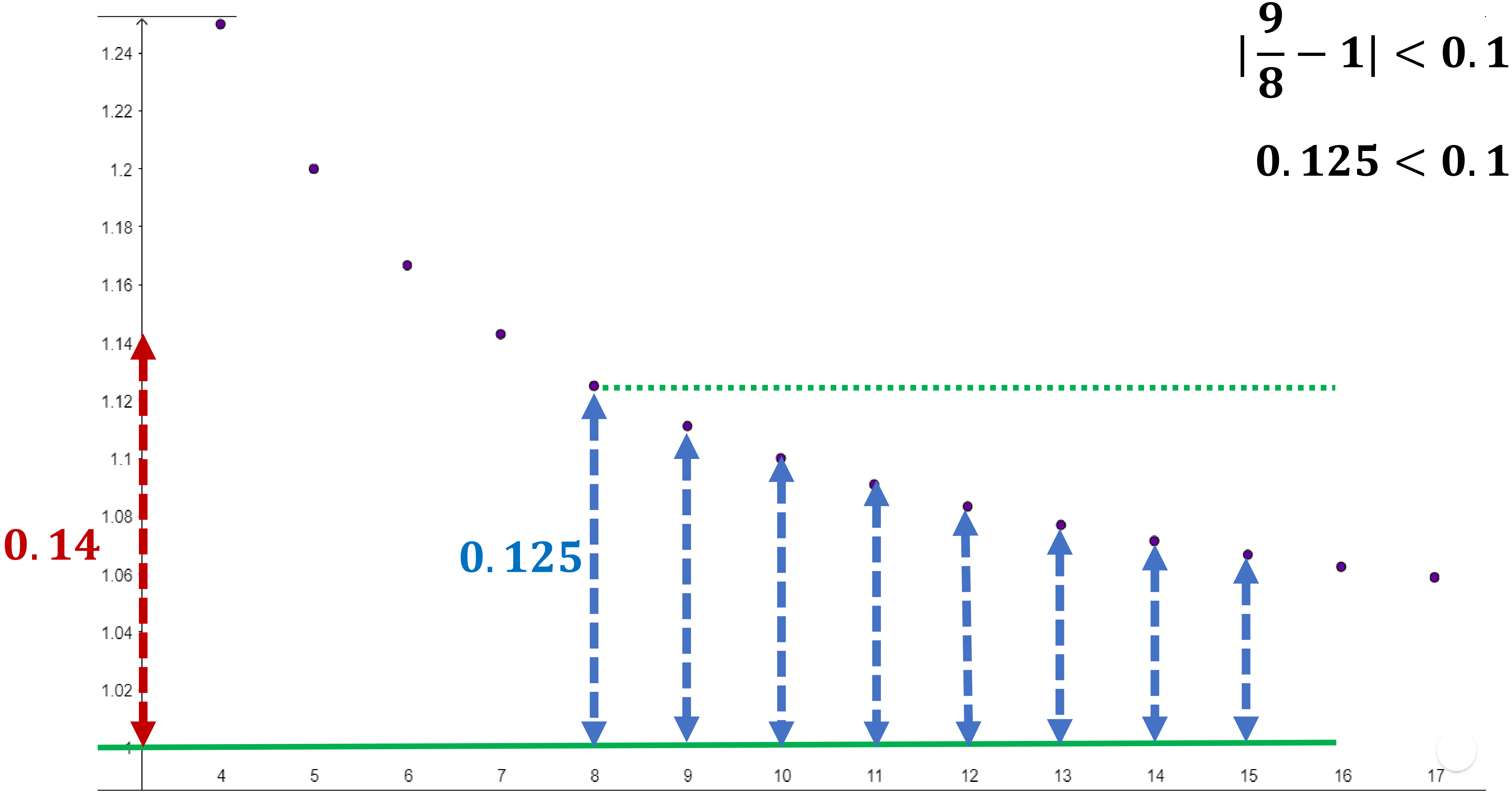
$$\boldsymbol{\varepsilon}=\mathbf{0.015}\Rightarrow\ \frac{\mathbf{1}}{\boldsymbol{\varepsilon}}\approx\mathbf{66.6}\qquad\qquad\boldsymbol{n}_{\mathbf{0}}\geq\mathbf{67}$$

$$1.\, b)\ \lim_{n\to\infty} \frac{n+1}{n} = 1 \qquad \varepsilon = 0.14 \qquad n_0 = 8$$

$$\left|\frac{n_0+1}{n_0} - 1\right| < 0.14$$

$$\left|\frac{9}{8} - 1\right| < 0.14$$

$$0.125 < 0.14$$



$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = ?$$

$n \in \mathbb{N}$	1	2	3	4	5	$\longrightarrow \infty$
$\left(\frac{1}{2}\right)^n$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\longrightarrow 0$

$$\forall \varepsilon > 0 \ \ \exists n_0 \in \mathbb{N} \ \ \forall n \geq n_0: \ \left|\left(\frac{1}{2}\right)^n - 0\right| < \varepsilon$$

$$\varepsilon > 0: \quad \left.\begin{aligned} & n > \frac{1}{\varepsilon} \\ & \boxed{2^n > n} \end{aligned}\right\} \Rightarrow 2^n > n > \frac{1}{\varepsilon} \quad \Rightarrow \frac{1}{2^n} < \frac{1}{n} < \varepsilon$$

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = ?$$

$n \in \mathbb{N}$	1	2	3	4	5	$\longrightarrow \infty$
$\left(\frac{1}{2}\right)^n$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\longrightarrow 0$

$$\forall \varepsilon > 0 \;\; \exists n_0 \in \mathbb{N} \;\; \forall n \geq n_0 : \; \left|\left(\frac{1}{2}\right)^n - 0\right| < \varepsilon$$

$$\varepsilon > 0: \; \left|\left(\frac{1}{2}\right)^n - 0\right| < \varepsilon \Leftrightarrow \left|\left(\frac{1}{2}\right)^n\right| < \varepsilon \Leftrightarrow \left(\frac{1}{2}\right)^n < \varepsilon \Leftrightarrow 2^{-n} < \varepsilon \Leftrightarrow \log_2(2^{-n}) < \log_2 \varepsilon$$

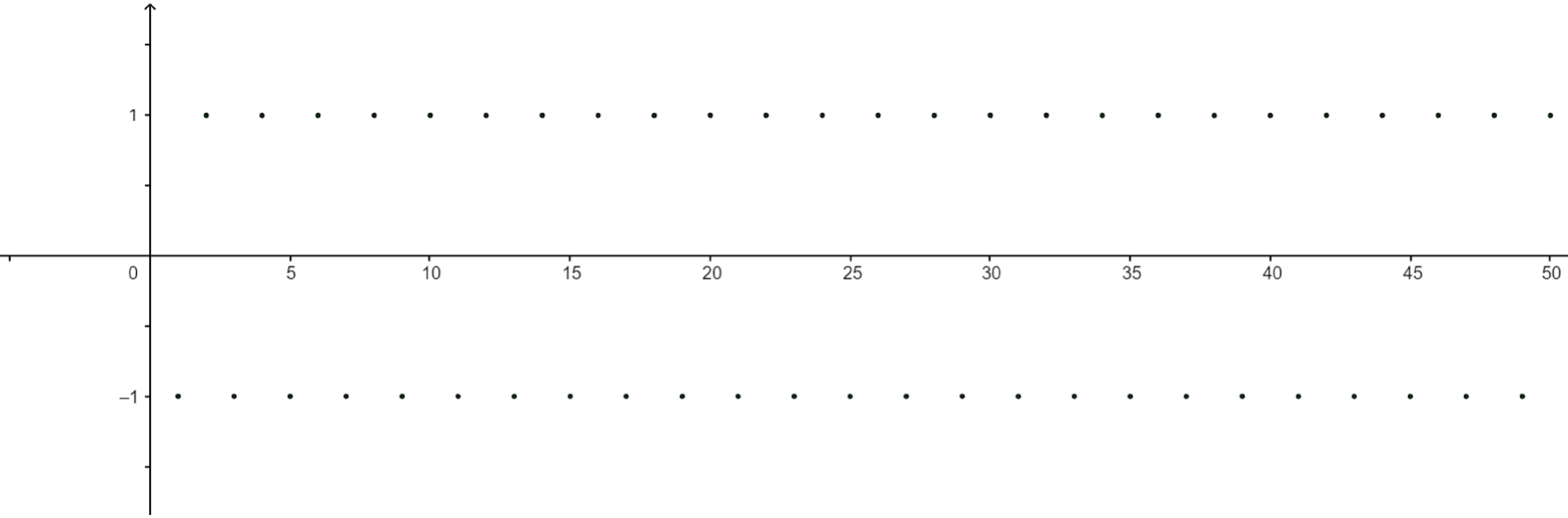
$$\Leftrightarrow -n < \log_2 \varepsilon \quad \Leftrightarrow n > -\log_2 \varepsilon \qquad \boxed{n_0 \in \mathbb{N} \wedge n_0 > -\log_2 \varepsilon}$$

$$\varepsilon = 0.0001 \Rightarrow -\log_2 \varepsilon = -\log_2(0.0001) \approx 13.288 \qquad n_0 \geq 14$$

$$\varepsilon = 0.000001 \Rightarrow -\log_2 \varepsilon = -\log_2(0.00001) \approx 19.93 \qquad n_0 \geq 20$$

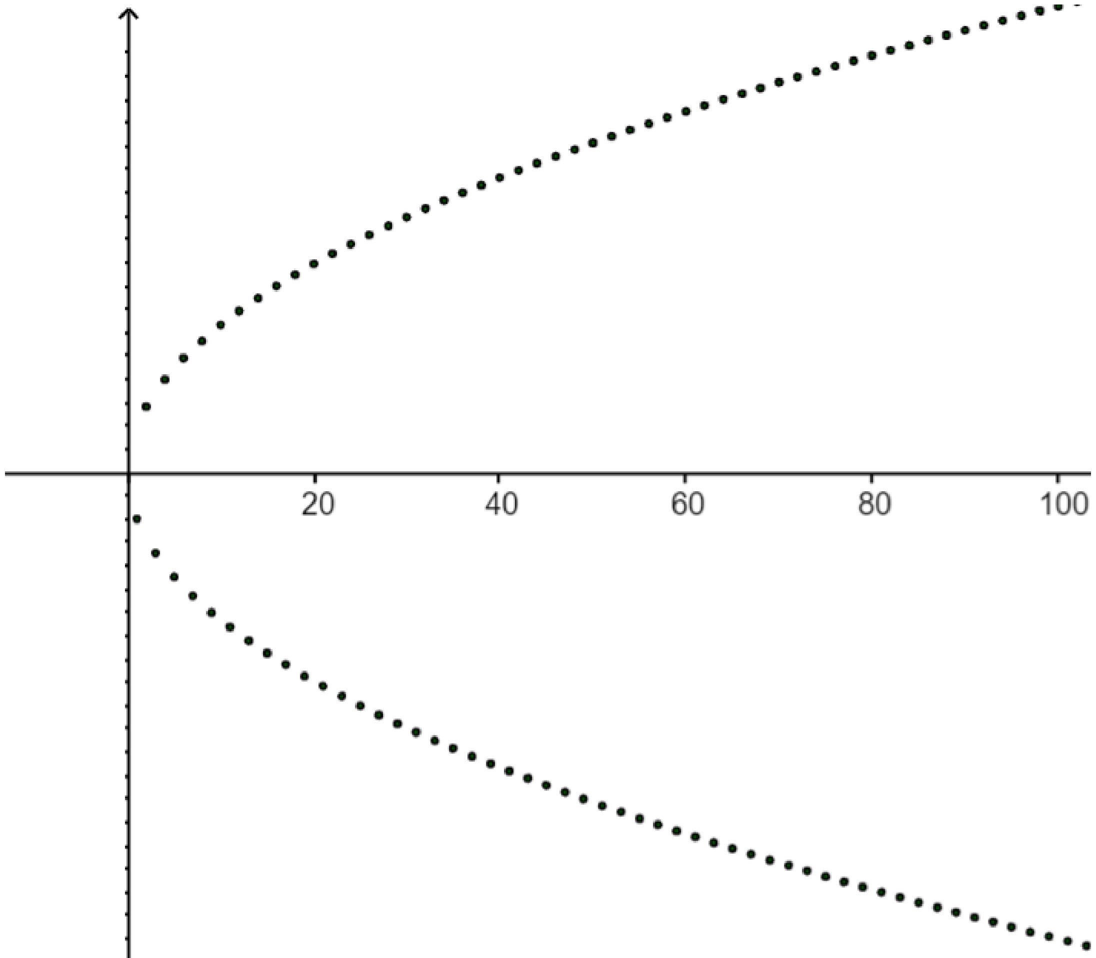
$$\lim_{n\to\infty}(-\mathbf{1})^{n} = ?$$

$n \in \mathbb{N}$	**1**	**2**	**3**	**4**	**5**	$\longrightarrow$	∞
$(-\mathbf{1})^{n}$	$-\mathbf{1}$	$\mathbf{1}$	$-\mathbf{1}$	$\mathbf{1}$	$-\mathbf{1}$	$\longrightarrow$	**neexistuje**



$$\lim_{n\to\infty} \cos(\pi n)\sqrt{n} = ?$$

$n \in \mathbb{N}$	**1**	**2**	**3**	**4**	**5**	$\longrightarrow$	∞
$\cos(\pi n)\sqrt{n}$	-1	$\sqrt{2}$	$-\sqrt{3}$	2	$-\sqrt{5}$	$\longrightarrow$	**neexistuje**



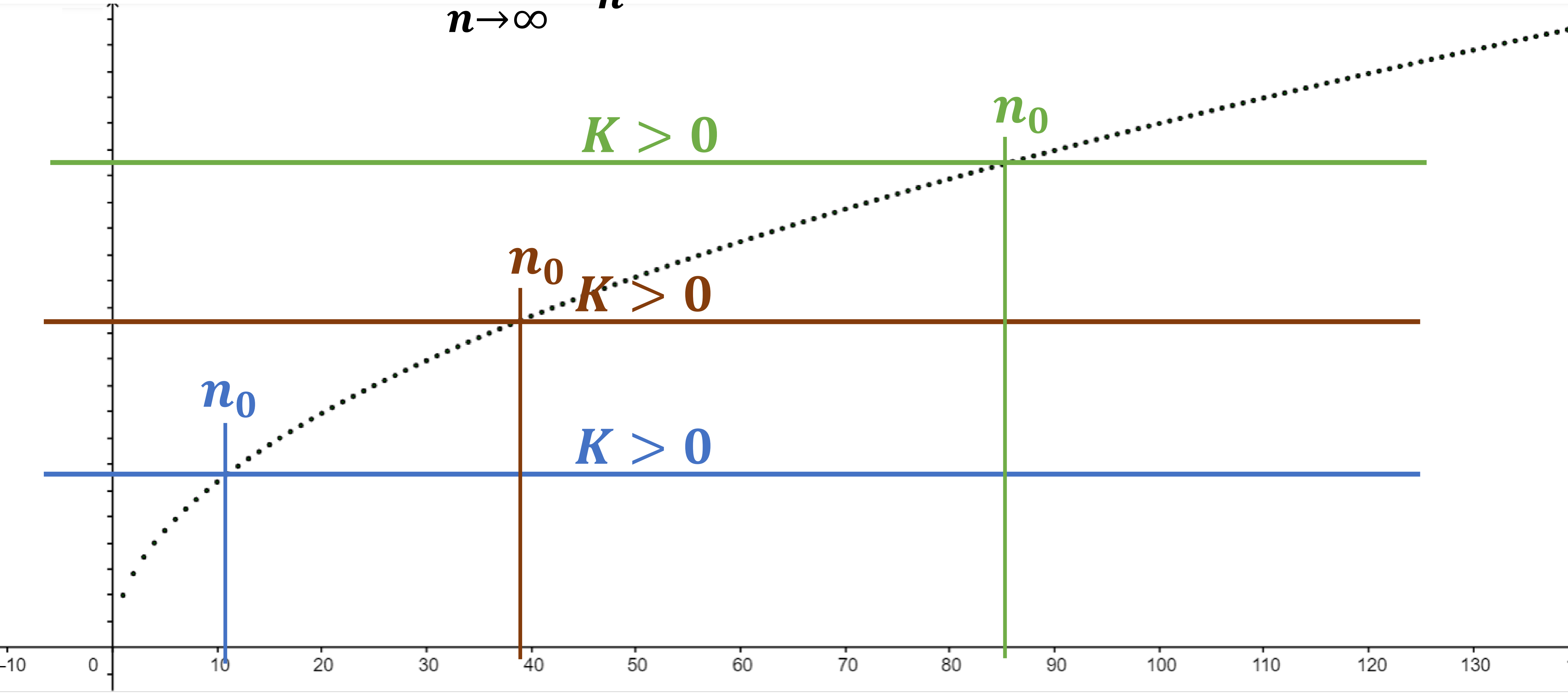
Definice (Divergentní posloupnost). Řekneme, že posloupnost $\{a_n\}_{n=1}^{\infty}$ diverguje do ∞ a píšeme

$\lim_{n\to\infty} a_n = \infty$ Jestliže platí:

$$\forall \boldsymbol{K} > \mathbf{0} \ \exists \boldsymbol{n_0} \in \mathbb{N} \ \forall \boldsymbol{n} > \boldsymbol{n_0}\colon |\boldsymbol{a_n}| > \boldsymbol{K}$$

$$\forall K > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : |a_n| > K$$

$$\lim_{n \to \infty} a_n = \infty$$



$$\lim_{n\to\infty}\frac{n^2}{10000n+1} = \frac{\infty}{\infty}$$

$$\lim_{n\to\infty}\frac{n^2}{10000n+1} = \lim_{n\to\infty}\frac{n^2}{n(10000+\frac{1}{n})} = \lim_{n\to\infty}\frac{n}{10000+\frac{1}{n}}$$

$$= \frac{\infty}{10000+0}$$

$$= \infty$$

$\mathbf{3.}\,\boldsymbol{d})\ \lim_{n\to\infty} \sqrt{n} - n \quad \infty - \infty$

$$\lim_{n\to\infty} \sqrt{n} - n = \lim_{n\to\infty} n\left(\frac{\sqrt{n}}{n} - 1\right) = \lim_{n\to\infty} n\left(\frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\sqrt{n}} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{1}{\sqrt{n}} - 1\right)$$

$$= \infty.(0 - 1)$$

$$= \infty.(-1)$$

$$= -\infty$$

$$\mathbf{2.}\, c)\ \lim_{n\to\infty} e^n = ?$$

$n \in \mathbb{N}$	1	2	3	4	5	$\longrightarrow$	∞
n	2.71	7.38	20.08	54.59	148.41	$\longrightarrow$	∞

$$\forall K > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : e^n > K$$

$$K > 0: \quad e^n > K \quad \Leftrightarrow \ln e^n > \ln K \quad \Leftrightarrow n > \ln K \qquad \boxed{n_0 \in \mathbb{N} \wedge n_0 > \ln K}$$

$$K = 100: \quad \ln K = \ln 100 = 4.605 \qquad n_0 \geq 5$$

$$K = 2000000: \quad \ln K = \ln 2000000 = 14.508 \qquad n_0 \geq 15$$

3(g) $\lim\limits_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\sqrt{n}$ $\quad(\infty-\infty)\infty$

$$\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\sqrt{n} = \lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\,\frac{(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})}(\sqrt{n})$$

$$= \lim_{n\to\infty}\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{(n+1-n)}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n\to\infty}\frac{\cancel{\sqrt{n}}}{(\frac{\sqrt{n+1}}{\sqrt{n}}+1)} = \lim_{n\to\infty}\frac{1}{\frac{\sqrt{n+1}}{\sqrt{n}}+1} = \lim_{n\to\infty}\frac{1}{\sqrt{\frac{n+1}{n}}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$= \frac{1}{\sqrt{1+0}} = \frac{1}{2}$$

$$3(h)\ \lim_{n\to\infty}\left(\frac{1+2+3+\cdots+n}{2+n}-\frac{n}{2}\right)$$

$$\frac{\infty}{\infty}-\infty$$

$$\boxed{1+2+3+\cdots+n=\frac{n(n+1)}{2}}$$

$$\begin{array}{lllllll} & 1+ & 2 & +\cdots & +(n-1) & +\ n & =A \\ + & n+ & (n-1) & +\cdots & +\ 2 & +1 & =A \\ \hline & (n+1) & +(n+1) & +\cdots & +(n+1) & +(n+1) & =2A \end{array}$$

$$n(n+1)=2A$$

$$\frac{n(n+1)}{2}=A$$

3(h) $\lim_{n\to\infty}\left(\frac{1+2+3+\cdots+n}{2+n}-\frac{n}{2}\right)$ $\frac{\infty}{\infty}-\infty$

$$\boxed{1+2+3+\cdots+n=\frac{n(n+1)}{2}}$$

$$\lim_{n\to\infty}\left(\frac{1+2+3+\cdots+n}{2+n}-\frac{n}{2}\right)=\lim_{n\to\infty}\left(\frac{\frac{n(n+1)}{2}}{2+n}-\frac{n}{2}\right)=\lim_{n\to\infty}\left(\frac{\frac{n^2+n}{2}}{2+n}-\frac{n}{2}\right)$$

$$=\lim_{n\to\infty}\left(\frac{n^2+n}{2(2+n)}-\frac{n}{2}\right)=\lim_{n\to\infty}\frac{n^2+n-n(2+n)}{2(2+n)}=\lim_{n\to\infty}\frac{-n}{4+2n}$$

$$=\lim_{n\to\infty}\frac{-\cancel{n}}{\cancel{n}(\frac{4}{n}+2)}=\lim_{n\to\infty}\frac{-1}{\frac{4}{n}+2}=\frac{-1}{+2}=-\frac{1}{2}$$

3(i) $\lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{n^3}$ $\frac{\infty}{\infty}$

$$1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\lim_{n\to\infty} \frac{\frac{n(n+1)(2n+1)}{6}}{n^3} = \lim_{n\to\infty} \frac{\frac{2n^3+3n^2+n}{6}}{n^3} = \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)}{6\cancel{n^3}} = \lim_{n\to\infty} \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}$$

$$= \frac{2+0+0}{6} = \frac{1}{3}$$

Věta (Omezená krát nulová). Nechť $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ jsou posloupnosti splňující:

(a) $\lim_{n\to\infty} a_n = 0$ (nulová)

(b) Od jistého indexu n_0 platí: $|b_n| < K \in \mathbb{R}$ (omezená)

Pak platí: $\lim_{n\to\infty} a_n \cdot b_n = 0.$

Věta (O dvou policajtech - **2P**).

Nechť $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ jsou posloupnosti splňující:

(a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = b \in \mathbb{R}$

(b) Od jistého indexu n_0 platí: $a_n < b_n < c_n$

Pak je posloupnost $\{b_n\}_{n=1}^{\infty}$ konvergentní a platí: $\lim_{n\to\infty} b_n = b$.

$$4(e)\ \lim_{n\to\infty}\frac{\sqrt[3]{n}\sin(n!)}{n+1}=?$$

$$\lim_{n\to\infty}\left(\frac{\sqrt[3]{n}}{n+1}\times\sin(n!)\right)=\lim_{n\to\infty}\left(\frac{\cancel{n^{\frac{1}{3}}}}{\cancel{n^{\frac{1}{3}}}\left(n^{\frac{2}{3}}+n^{-\frac{1}{3}}\right)}\times\sin(n!)\right)=\lim_{n\to\infty}\left(\underbrace{\frac{1}{n^{\frac{2}{3}}+n^{-\frac{1}{3}}}}_{\text{Nulová}}\times\underbrace{\sin(n!)}_{\text{Omezená}}\right)$$

$$=0$$

$$\lim_{n\to\infty}\frac{1}{n^{\frac{2}{3}}+n^{-\frac{1}{3}}}=\frac{1}{\infty+0}=0$$

$$4(e)\ \lim_{n\to\infty}\frac{\sqrt[3]{n}\sin(n!)}{n+1} \qquad -1\le\sin(n!)\le 1, \qquad \frac{\sqrt[3]{n}}{n+1}>0$$

$$-\frac{\sqrt[3]{n}}{n+1}\le\frac{\sqrt[3]{n}\sin(n!)}{n+1}\le\frac{\sqrt[3]{n}}{n+1}$$

$$\lim_{n\to\infty}\left(-\frac{\sqrt[3]{n}}{n+1}\right)=\lim_{n\to\infty}\left(-\frac{\cancel{n^{\frac{1}{3}}}}{\cancel{n^{\frac{1}{3}}}\left(n^{\frac{2}{3}}+n^{-\frac{1}{3}}\right)}\right)=\lim_{n\to\infty}\left(-\frac{1}{n^{\frac{2}{3}}+n^{-\frac{1}{3}}}\right)=-\frac{1}{\infty+0}=0$$

$$\lim_{n\to\infty}\frac{\sqrt[3]{n}}{n+1}=\lim_{n\to\infty}\frac{\cancel{n^{\frac{1}{3}}}}{\cancel{n^{\frac{1}{3}}}\left(n^{\frac{2}{3}}+n^{-\frac{1}{3}}\right)}=\lim_{n\to\infty}\frac{1}{n^{\frac{2}{3}}+n^{-\frac{1}{3}}}=\frac{1}{\infty+0}=0$$

$$\lim_{n\to\infty}\left(\frac{\sin(n)}{n} - \sqrt[3]{n}\right) \qquad = \lim_{n\to\infty}\frac{\sin(n)}{n} - \lim_{n\to\infty}\sqrt[3]{n}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sin(n) - \lim_{n\to\infty}\sqrt[3]{n} \quad = 0 - \infty = -\infty$$

Nulová . Omezená = Nulová

$$\mathbf{4}(\boldsymbol{o})\ \lim_{n\to\infty} \sqrt[n]{\mathbf{2}^n + \mathbf{4}^n} = ?$$

$$4 = \sqrt[n]{4^n} \le \sqrt[n]{2^n + 4^n} \le \sqrt[n]{4^n + 4^n} = \sqrt[n]{2 \times 4^n} = 4.$$

$$4 \le \sqrt[n]{2^n + 4^n} \le 4.\sqrt[n]{2}$$

$$\left.\begin{array}{l} \lim\limits_{n\to\infty} 4 = 4 \\ \\ \lim\limits_{n\to\infty} 4.\sqrt[n]{2} = \lim\limits_{n\to\infty} 4 \times \lim\limits_{n\to\infty} \sqrt[n]{2} = 4 \times 1 = 4 \end{array}\right\} \lim_{n\to\infty} \sqrt[n]{2^n + 4^n} = 4$$

Tradičním nástrojem při řešení limit s odmocninou je následující vzorce:

$$A^n - B^n = (A - B)(A^{n-1}B^0 + A^{n-2}B^1 + \cdots + A^1B^{n-2} + A^0B^{n-1})$$

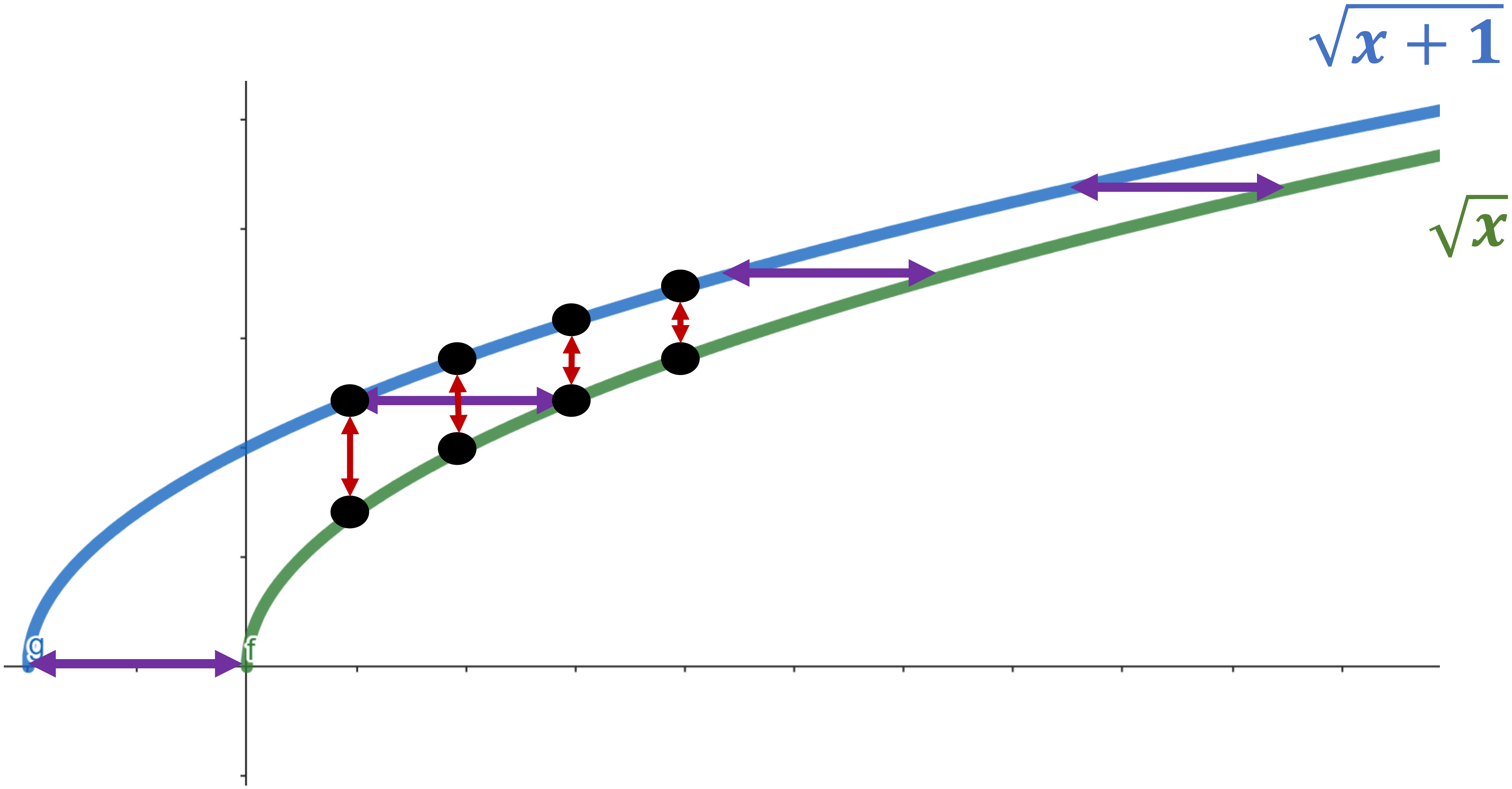
$$A^2 - B^2 = (A - B)(A + B)$$

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

$$\underbrace{(\sqrt[3]{n^2+11}}_{A} - \underbrace{\sqrt[3]{n^2+1})}_{B} \; (\underbrace{\left(\sqrt[3]{n^2+11}\right)^2}_{A^2} + \underbrace{(\sqrt[3]{n^2+11})(\sqrt[3]{n^2+1})}_{A.B} + \underbrace{(\sqrt[3]{n^2+1})^2}_{B^2})$$

$$= \underbrace{\left(\sqrt[3]{n^2+11}\right)^3}_{A^3} - \underbrace{\left(\sqrt[3]{n^2+1}\right)^3}_{B^3} = n^2 + 11 - (n^2 + 1) = 10$$

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$$



$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) \qquad \infty-\infty$$

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)\frac{\left(\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)} = \lim_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$$

$$\lim_{n\to\infty}(\sqrt[3]{n^3+2n^2}-\sqrt[3]{n^3-n^2+1})$$

$$\sqrt[3]{n^3+2n^2}=A \qquad \sqrt[3]{n^3-n^2+1}=B$$

$$(A-B)\left(A^2+AB+B^2\right)=A^3-B^3$$

$$\lim_{n\to\infty}(\sqrt[3]{n^3+2n^2}-\sqrt[3]{n^3-n^2+1})\frac{(\left(\sqrt[3]{n^3+2n^2}\right)^2+\left(\sqrt[3]{n^3+2n^2}\right)\left(\sqrt[3]{n^3-n^2+1}\right)+\left(\sqrt[3]{n^3-n^2+1}\right)^2)}{\left(\sqrt[3]{n^3+2n^2}\right)^2+\left(\sqrt[3]{n^3+2n^2}\right)\left(\sqrt[3]{n^3-n^2+1}\right)+\left(\sqrt[3]{n^3-n^2+1}\right)^2}$$

$$=\lim_{n\to\infty}\frac{n^3+2n^2-(n^3-n^2+1)}{(n^3+2n^2)^{\frac{2}{3}}+(n^3+2n^2)^{\frac{1}{3}}(n^3-n^2+1)^{\frac{1}{3}}+(n^3-n^2+1)^{\frac{2}{3}}}$$

$$=\lim_{n\to\infty}\frac{3n^2-1 \longrightarrow \cancel{n^2}(3-\frac{1}{n^2})}{n^2(1+\frac{2}{n})^{\frac{2}{3}}+n(1+\frac{2}{n})^{\frac{1}{3}}n(1-\frac{1}{n}+\frac{1}{n^3})^{\frac{1}{3}}+n^2(1-\frac{1}{n}+\frac{1}{n^3})^{\frac{2}{3}}}=\frac{3-0}{1+1+1}=1$$

$$\downarrow$$

$$\cancel{n^2}((1+\frac{2}{n})^{\frac{2}{3}}+(1+\frac{2}{n})^{\frac{1}{3}}(1-\frac{1}{n}+\frac{1}{n^3})^{\frac{1}{3}}+(1-\frac{1}{n}+\frac{1}{n^3})^{\frac{2}{3}})$$

$\boxed{4(p)}$

$$\lim_{n\to\infty}\frac{\sqrt[3]{n^2+11}-\sqrt[3]{n^2+1}}{\sqrt[3]{n^2+6}-\sqrt[3]{n^2}}=\lim_{n\to\infty}\left(\left(\sqrt[3]{n^2+11}-\sqrt[3]{n^2+1}\right)\times\frac{1}{\sqrt[3]{n^2+6}-\sqrt[3]{n^2}}\right)=$$

$$\lim_{n\to\infty}\left(\frac{\left(\sqrt[3]{n^2+11}-\sqrt[3]{n^2+1}\right)\times\left(\left(\sqrt[3]{n^2+11}\right)^2+\left(\sqrt[3]{n^2+11}\right)\left(\sqrt[3]{n^2+1}\right)+\left(\sqrt[3]{n^2+1}\right)^2\right)}{\left(\sqrt[3]{n^2+11}\right)^2+\left(\sqrt[3]{n^2+11}\right)\left(\sqrt[3]{n^2+1}\right)+\left(\sqrt[3]{n^2+1}\right)^2}\right.$$

$$\left.\times\frac{\left(\sqrt[3]{n^2+6}\right)^2+\left(\sqrt[3]{n^2+6}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2}{\left(\sqrt[3]{n^2+6}-\sqrt[3]{n^2}\right)\times\left(\left(\sqrt[3]{n^2+6}\right)^2+\left(\sqrt[3]{n^2+6}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2\right)}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\left(\sqrt[3]{n^2+11}\right)^3-\left(\sqrt[3]{n^2+1}\right)^3}{\left(\sqrt[3]{n^2+11}\right)^2+\left(\sqrt[3]{n^2+11}\right)\left(\sqrt[3]{n^2+1}\right)+\left(\sqrt[3]{n^2+1}\right)^2}\times\frac{\left(\sqrt[3]{n^2+6}\right)^2+\left(\sqrt[3]{n^2+6}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2}{\left(\sqrt[3]{n^2+6}\right)^3-\left(\sqrt[3]{n^2}\right)^3}\right)$$

$$=\lim_{n\to\infty}\left(\frac{n^2+11-(n^2+1)}{\left(\sqrt[3]{n^2+11}\right)^2+\left(\sqrt[3]{n^2+11}\right)\left(\sqrt[3]{n^2+1}\right)+\left(\sqrt[3]{n^2+1}\right)^2}\times\frac{\left(\sqrt[3]{n^2+6}\right)^2+\left(\sqrt[3]{n^2+6}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2}{n^2+6-n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n^2+11-(n^2+1)}{\left(\sqrt[3]{n^2+11}\right)^2+\left(\sqrt[3]{n^2+11}\right)\left(\sqrt[3]{n^2+1}\right)+\left(\sqrt[3]{n^2+1}\right)^2}\times\frac{\left(\sqrt[3]{n^2+6}\right)^2+\left(\sqrt[3]{n^2+6}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2}{n^2+6-n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{10}{6}\times\frac{\left(\sqrt[3]{n^2+6}\right)^2+\left(\sqrt[3]{n^2+6}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2}{\left(\sqrt[3]{n^2+11}\right)^2+\left(\sqrt[3]{n^2+11}\right)\left(\sqrt[3]{n^2+1}\right)+\left(\sqrt[3]{n^2+1}\right)^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{5}{3}\times\frac{\left(\sqrt[3]{n^2(1+\frac{6}{n^2})}\right)^2+\left(\sqrt[3]{n^2(1+\frac{6}{n^2})}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2}{\left(\sqrt[3]{n^2(1+\frac{11}{n^2})}\right)^2+\left(\sqrt[3]{n^2(1+\frac{11}{n^2})}\right)\left(\sqrt[3]{n^2(1+\frac{1}{n^2})}\right)+\left(\sqrt[3]{n^2(1+\frac{1}{n^2})}\right)^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{5}{3}\times\frac{\left(\sqrt[3]{n^2\left(1+\frac{6}{n^2}\right)}\right)^2+\left(\sqrt[3]{n^2\left(1+\frac{6}{n^2}\right)}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2}{\left(\sqrt[3]{n^2\left(1+\frac{11}{n^2}\right)}\right)^2+\left(\sqrt[3]{n^2\left(1+\frac{11}{n^2}\right)}\right)\left(\sqrt[3]{n^2\left(1+\frac{1}{n^2}\right)}\right)+\left(\sqrt[3]{n^2\left(1+\frac{1}{n^2}\right)}\right)^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{5}{3}\times\frac{\left(\sqrt[3]{n^2}\right)^2\left(\sqrt[3]{\left(1+\frac{6}{n^2}\right)}\right)^2+\left(\sqrt[3]{n^2}\right)\left(\sqrt[3]{\left(1+\frac{6}{n^2}\right)}\right)\left(\sqrt[3]{n^2}\right)+\left(\sqrt[3]{n^2}\right)^2}{\left(\sqrt[3]{n^2}\right)^2\left(\sqrt[3]{\left(1+\frac{11}{n^2}\right)}\right)^2+\left(\sqrt[3]{n^2}\right)\left(\sqrt[3]{\left(1+\frac{11}{n^2}\right)}\right)\left(\sqrt[3]{n^2}\right)\left(\sqrt[3]{\left(1+\frac{1}{n^2}\right)}\right)+\left(\sqrt[3]{n^2}\right)^2\left(\sqrt[3]{\left(1+\frac{1}{n^2}\right)}\right)^2}\right.$$

$$= \lim_{n\to\infty}\left(\frac{5}{3}\times\frac{\left(\sqrt[3]{n^2}\right)^2\left(\left(\sqrt[3]{\left(1+\frac{6}{n^2}\right)}\right)^2+\left(\sqrt[3]{\left(1+\frac{6}{n^2}\right)}\right)+1\right)}{\left(\sqrt[3]{n^2}\right)^2\left(\left(\sqrt[3]{\left(1+\frac{11}{n^2}\right)}\right)^2+\left(\sqrt[3]{\left(1+\frac{11}{n^2}\right)}\right)\left(\sqrt[3]{\left(1+\frac{1}{n^2}\right)}\right)+\left(\sqrt[3]{\left(1+\frac{1}{n^2}\right)}\right)^2\right)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{5}{3}\times\frac{\cancel{\left(\sqrt[3]{n^2}\right)^2}\left(\left(\sqrt[3]{\left(1+\frac{6}{n^2}\right)}\right)^2+\left(\sqrt[3]{\left(1+\frac{6}{n^2}\right)}\right)+1\right)}{\cancel{\left(\sqrt[3]{n^2}\right)^2}\left(\left(\sqrt[3]{\left(1+\frac{11}{n^2}\right)}\right)^2+\left(\sqrt[3]{\left(1+\frac{11}{n^2}\right)}\right)\left(\sqrt[3]{\left(1+\frac{1}{n^2}\right)}\right)+\left(\sqrt[3]{\left(1+\frac{1}{n^2}\right)}\right)^2\right)}\right)$$

$$= \lim_{n\to\infty}\frac{5}{3}\times\lim_{n\to\infty}\frac{\left(\sqrt[3]{\left(1+\frac{6}{n^2}\right)}\right)^2+\left(\sqrt[3]{\left(1+\frac{6}{n^2}\right)}\right)+1}{\left(\sqrt[3]{\left(1+\frac{11}{n^2}\right)}\right)^2+\left(\sqrt[3]{\left(1+\frac{11}{n^2}\right)}\right)\left(\sqrt[3]{\left(1+\frac{1}{n^2}\right)}\right)+\left(\sqrt[3]{\left(1+\frac{1}{n^2}\right)}\right)^2}$$

$$= \quad\times\frac{\left(\sqrt[3]{(1+0)}\right)^2+\left(\sqrt[3]{(1+0)}\right)+1}{\left(\sqrt[3]{(1+0)}\right)^2+\left(\sqrt[3]{(1+0)}\right)\left(\sqrt[3]{(1+0)}\right)+\left(\sqrt[3]{(1+0)}\right)^2}=\frac{5}{3}\times\frac{1+1+1}{1+1+1}=\frac{5}{3}\times\frac{3}{3}=\frac{5}{3}$$

$\boxed{\mathbf{4}(\boldsymbol{q})}$

$$\lim_{n\to\infty} \frac{\sin \boldsymbol{n} - \cos n^2}{\boldsymbol{n}^2 + \mathbf{17}} = \lim_{n\to\infty} \left(\frac{\mathbf{1}}{\boldsymbol{n}^2 + \mathbf{17}}\right)(\sin \boldsymbol{n} - \cos n^2)$$

Nulová . Omezená = Nulová

$\boxed{4(r)}$

$$\lim_{n\to\infty}\left(\sqrt[3]{n^3+7n+1}-\sqrt{n^2+1}\right)n$$

$$\sqrt[3]{n^3+7n+1}=A \qquad \sqrt{n^2+1}=B$$

$$\boxed{n^2+1}$$

$$(A-B)(A+B)=A^2-B^2 \qquad A^2-B^2=\left(\sqrt[3]{n^3+7n+1}\right)^2-\left(\sqrt{n^2+1}\right)^2$$

$$(A-B)\left(A^2+AB+A^2\right)=A^3-B^3 \qquad A^3-B^3=\left(\sqrt[3]{n^3+7n+1}\right)^3-\left(\sqrt{n^2+1}\right)^3$$

$$\boxed{n^3+7n+1}$$

$$(A-B)\left(A^5+A^4B+A^3B^2+A^2B^3+AB^4+B^5\right)=A^6-B^6$$

$$A^6-B^6=\left(\sqrt[3]{n^3+7n+1}\right)^6-\left(\sqrt{n^2+1}\right)^6=(n^3+7n+)^2-(n^2+1)^3$$

$\boxed{4(r)}$

$$\lim_{n\to\infty}\left(\sqrt[3]{n^3+7n+1}-\sqrt{n^2+1}\right)n \qquad \sqrt[3]{n^3+7n+1}=A \qquad \sqrt{n^2+1}=B$$

$$\lim_{n\to\infty}[(\sqrt[3]{n^3+7n+1}-\sqrt{n^2+1})\times\frac{A^5+A^4B+\cdots+B^5}{A^5+A^4B+\cdots+B^5}\,n]$$

$$\lim_{n\to\infty}\frac{(\sqrt[3]{n^3+7n+1})^6-(\sqrt{n^2+1})^6}{\left(\sqrt[3]{n^3+7n+1}\right)^5+\left(\sqrt[3]{n^3+7n+1}\right)^4\left(\sqrt{n^2+1}\right)^1+\cdots+\left(\sqrt{n^2+1}\right)^5}\,n$$

$$\lim_{n\to\infty}\frac{(n^3+7n+1)^2-(n^2+1)^3}{\left(\sqrt[3]{n^3(1+\frac{7}{n^2}+\frac{1}{n^3})}\right)^5+\left(\sqrt[3]{n^3(1+\frac{7}{n^2}+\frac{1}{n^3})}\right)^4\left(\sqrt{n^2(1+\frac{1}{n^2})}\right)^1+\cdots+\left(\sqrt{n^2(1+\frac{1}{n^2})}\right)^5}\,n$$

$$\lim_{n\to\infty}\frac{11n^5+2n^4+46n^3+14n^2}{n^5\left(\sqrt[3]{1+\frac{7}{n^2}+\frac{1}{n^3}}\right)^5+n^5\left(\sqrt[3]{1+\frac{7}{n^2}+\frac{1}{n^3}}\right)^4\left(\sqrt{1+\frac{1}{n^2}}\right)^1+\cdots+n^5\left(\sqrt{1+\frac{1}{n^2}}\right)^5}$$

$$\lim_{n\to\infty} \frac{11n^5 + 2n^4 + 46n^3 + 14n^2}{n^5\left(\sqrt[3]{1+\frac{7}{n^2}+\frac{1}{n^3}}\right)^5 + n^5\left(\sqrt[3]{1+\frac{7}{n^2}+\frac{1}{n^3}}\right)^4\left(\sqrt{1+\frac{1}{n^2}}\right)^1 + \cdots + n^5\left(\sqrt{1+\frac{1}{n^2}}\right)^5}$$

$$\lim_{n\to\infty} \frac{n^5(11 + \frac{2}{n} + \frac{46}{n^2} + \frac{14}{n^3})}{n^5\left(\sqrt[3]{1+\frac{7}{n^2}+\frac{1}{n^3}}\right)^5 + n^5\left(\sqrt[3]{1+\frac{7}{n^2}+\frac{1}{n^3}}\right)^4\left(\sqrt{1+\frac{1}{n^2}}\right)^1 + \cdots + n^5\left(\sqrt{1+\frac{1}{n^2}}\right)^5}$$

$$= \frac{11+0+0+0}{6} = \frac{11}{6}$$

Věta (Růstová škála). Od jisté hodnoty n platí

$$\log(\log(\boldsymbol{n})) \ll \mathbf{1000} \log \boldsymbol{n} \ll \boldsymbol{n}^{\frac{\mathbf{1}}{\mathbf{1000}}} \ll \boldsymbol{n} \ll \boldsymbol{n}^{\mathbf{1000}} \ll \mathbf{1{,}0001}^{\boldsymbol{n}} \ll \boldsymbol{n}! \ll \boldsymbol{n}^{\boldsymbol{n}}$$

Kde zápis $a_n \ll b_n$ znamená „a_n roste výrazně pomaleji než b_n", tedy:

$$\lim_{\boldsymbol{n}\to\infty} \frac{\boldsymbol{a_n}}{\boldsymbol{b_n}} = \mathbf{0} \qquad \text{a} \qquad \lim_{\boldsymbol{n}\to\infty} \frac{\boldsymbol{b_n}}{\boldsymbol{a_n}} = \infty$$

Poučka. Stejně jako při práci s polynomy je vhodné vytýkat nejrychleji rostoucí výraz.

$$\lim_{n\to\infty} \frac{n}{2^n} = ?$$

$$\log(\log(n)) \ll 1000 \log n \ll n^{\frac{1}{1000}} \ll n \ll n^{1000} \ll 1{,}0001^n \ll n! \ll n^n$$

$$n \ll 2^n$$

$$\lim_{n\to\infty} \frac{n}{2^n} = 0$$

$$\lim_{n\to\infty} \frac{2^n}{n!}$$

$$\log(\log(n)) \ll 1000 \log n \ll n \ll n^{1000} \ll 1{,}0001^n \ll n! \ll n^n$$

$$2^n \ll n!$$

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0$$

$$\lim_{n\to\infty}\frac{\ln \boldsymbol{n}}{\sqrt{\boldsymbol{n}}}$$

$$\log(\log(\boldsymbol{n})) \ll \mathbf{1000}\log \boldsymbol{n} \ll \boldsymbol{n}^{\frac{1}{1000}} \ll \boldsymbol{n} \ll \boldsymbol{n}^{1000} \ll \mathbf{1{,}0001}^{\boldsymbol{n}} \ll \boldsymbol{n}! \ll \boldsymbol{n}^{\boldsymbol{n}}$$

$$\ln \boldsymbol{n} \ll \boldsymbol{n}^{\frac{1}{2}} = \sqrt{\boldsymbol{n}}$$

$$\lim_{n\to\infty}\frac{\ln \boldsymbol{n}}{\sqrt{\boldsymbol{n}}} = \mathbf{0}$$

$$\lim_{n\to\infty} \frac{n!}{n^n}$$

$$\log(\log(n)) \ll 1000\log n \ll n^{\frac{1}{1000}} \ll n \ll n^{1000} \ll 1{,}0001^n \ll \boxed{n! \ll n^n}$$

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

$$5(a)\ \lim_{n\to\infty}\frac{n+102\log n+n^2}{\sqrt{n}-n^{\frac{3}{2}}+2^n} = \lim_{n\to\infty}\frac{n^2(\frac{1}{n}+\frac{102\log n}{n^2}+1)}{2^n(\frac{\sqrt{n}}{2^n}-\frac{n^{\frac{3}{2}}}{2^n}+1)}$$

$$=\lim_{n\to\infty}\frac{n^2}{2^n}\ \lim_{n\to\infty}\frac{\frac{1}{n}+\frac{102\log n}{n^2}+1}{\frac{\sqrt{n}}{2^n}-\frac{n^{\frac{3}{2}}}{2^n}+1}$$

$$=0\times\frac{0+0+1}{0-0+1}=0$$

$\mathbf{5(b)}$

$$\lim_{n\to\infty}\frac{5^n+n^2}{\sqrt{n}-3^{2n}} = \lim_{n\to\infty}\frac{5^n\left(1+\frac{n^2}{5^n}\right)}{3^{2n}\left(\frac{\sqrt{n}}{3^{2n}}-1\right)} = \lim_{n\to\infty}\frac{5^n}{3^{2n}}\lim_{n\to\infty}\frac{1+\frac{n^2}{5^n}}{\frac{\sqrt{n}}{3^{2n}}-1}$$

$$= \lim_{n\to\infty}\frac{5^n}{9^n}\frac{\lim_{n\to\infty}\left(1+\frac{n^2}{5^n}\right)}{\lim_{n\to\infty}\left(\frac{\sqrt{n}}{3^{2n}}-1\right)}$$

$$= \lim_{n\to\infty}\left(\frac{5}{9}\right)^n\frac{\lim_{n\to\infty}\left(1+\frac{n^2}{5^n}\right)}{\lim_{n\to\infty}\left(\frac{\sqrt{n}}{3^{2n}}-1\right)} = (0)\left(\frac{1+0}{0-1}\right) = 0$$

$\mathbf{5(c)}$

$$\lim_{n\to\infty}\frac{\sqrt{4n-2\log n+11n^2}}{\sqrt{n}-n}=\lim_{n\to\infty}\frac{\sqrt{n^2(\frac{4}{n}-2\frac{\log n}{n^2}+11)}}{n(\frac{\sqrt{n}}{n}-1)}=\lim_{n\to\infty}\frac{n\times\sqrt{(\frac{4}{n}-2\frac{\log n}{n^2}+11)}}{n(\frac{\sqrt{n}}{n}-1)}$$

$$=\lim_{n\to\infty}\frac{\sqrt{(\frac{4}{n}-2\frac{\log n}{n^2}+11)}}{(\frac{\sqrt{n}}{n}-1)}=\frac{\lim_{n\to\infty}\sqrt{(\frac{4}{n}-2\frac{\log n}{n^2}+11)}}{\lim_{n\to\infty}(\frac{\sqrt{n}}{n}-1)}=\frac{\sqrt{\lim_{n\to\infty}(\frac{4}{n}-2\frac{\log n}{n^2}+11)}}{\lim_{n\to\infty}(\frac{\sqrt{n}}{n}-1)}$$

Pro $k\in\mathbb{N}$ platí:

$$\lim_{n\to\infty}\sqrt[k]{a_n}=\sqrt[k]{\lim_{n\to\infty}a_n}$$

$$\lim_{n\to\infty}{a_n}^k=(\lim_{n\to\infty}a_n)^k$$

$$=\frac{\sqrt{0-0+11}}{0-1}=-\sqrt{11}$$

5(d)

$$\lim_{n\to\infty}\frac{\sqrt[3]{4n-2\log n+11.27^n}}{(\frac{7}{2})^n+2^n} = \lim_{n\to\infty}\frac{\sqrt[3]{27^n(\frac{4n}{27^n}-\frac{2\log n}{27^n}+11)}}{(3,5)^n+2^n}$$

$$= \lim_{n\to\infty}\frac{(3^n)(\sqrt[3]{\frac{4n}{27^n}-\frac{2\log n}{27^n}+11}\,)}{(3,5)^n(1+\frac{2^n}{(3,5)^n})} = \lim_{n\to\infty}(\frac{3}{3,5})^n\frac{(\sqrt[3]{\frac{4n}{27^n}-\frac{2\log n}{27^n}+11}\,)}{(1+(\frac{2}{3,5})^n)}$$

$$= \lim_{n\to\infty}(\frac{3}{3,5})^n\lim_{n\to\infty}\frac{(\sqrt[3]{\frac{4n}{27^n}-\frac{2\log n}{27^n}+11}\,)}{(1+(\frac{2}{3,5})^n)} = \lim_{n\to\infty}(\frac{3}{3,5})^n\times\frac{\lim_{n\to\infty}\sqrt[3]{\frac{4n}{27^n}-\frac{2\log n}{27^n}+11}}{\lim_{n\to\infty}(1+(\frac{2}{3.5})^n)}$$

$$= \lim_{n\to\infty}(\frac{3}{3,5})^n\times\frac{\sqrt[3]{\lim_{n\to\infty}(\frac{4n}{27^n}-\frac{2\log n}{27^n}+11)}}{\lim_{n\to\infty}1+\lim_{n\to\infty}(\frac{2}{3.5})^n} = 0\times\frac{\sqrt[3]{11}}{1} = 0$$

$\mathbf{5(e)}$

$$\lim_{n\to\infty}\frac{1^n+2^n+3^n+4^n+5^n}{(5.0001)^n} = \lim_{n\to\infty}\frac{5^n((\frac{1}{5})^n+(\frac{2}{5})^n+(\frac{3}{5})^n+(\frac{4}{5})^n+1)}{(5.0001)^n}$$

$$= \lim_{n\to\infty}\left(\frac{5}{5.0001}\right)^n((\frac{1}{5})^n+(\frac{2}{5})^n+(\frac{3}{5})^n+(\frac{4}{5})^n+1)$$

$$= \lim_{n\to\infty}\left(\frac{5}{5.0001}\right)^n \lim_{n\to\infty}((\frac{1}{5})^n+(\frac{2}{5})^n+(\frac{3}{5})^n+(\frac{4}{5})^n+1)$$

$$= 0(0+0+0+0+1) \quad = 0$$

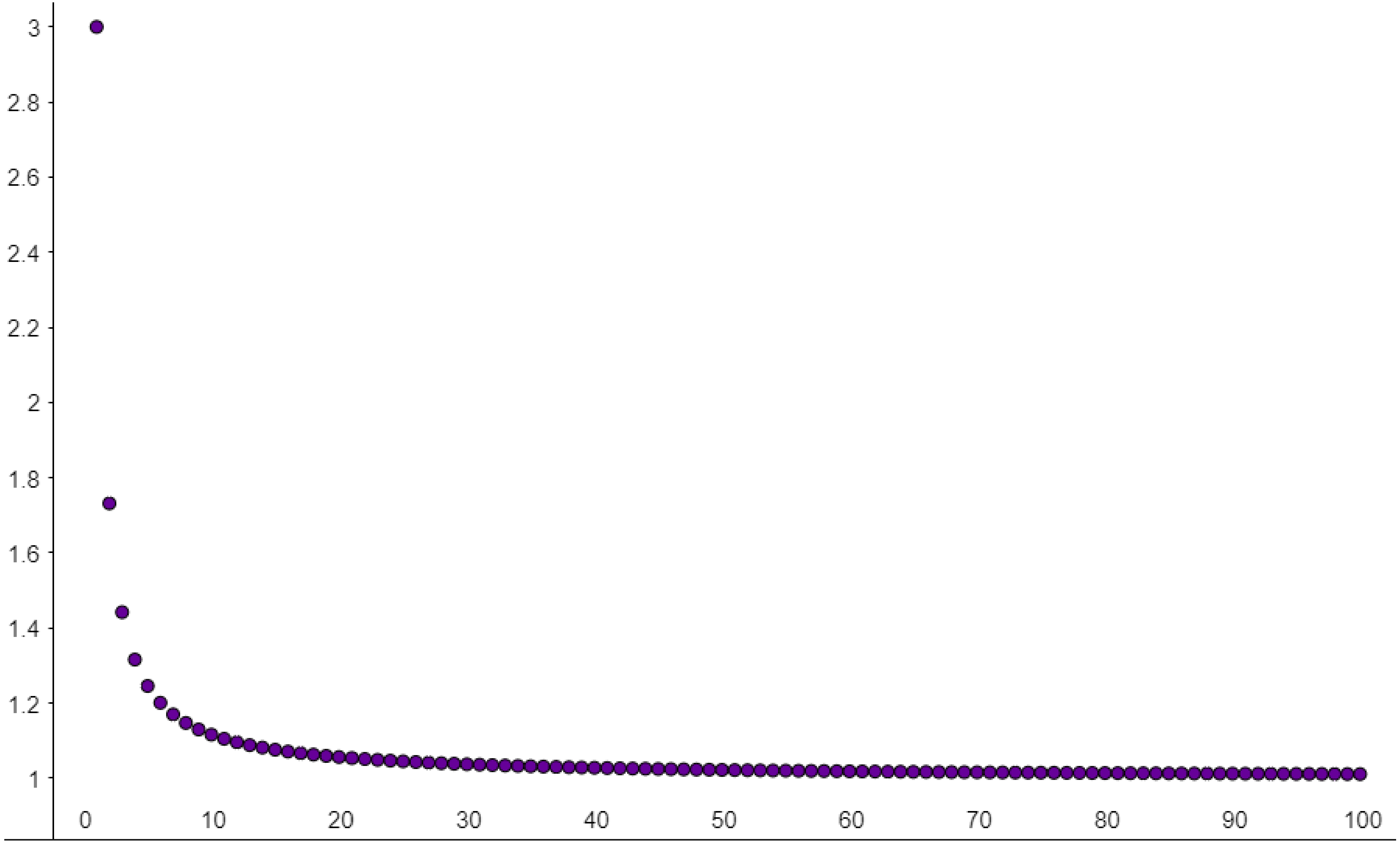
Věta (n-tá odmocnina)**.**

(a) Pro $c > 0$ $\lim_{n\to\infty} \sqrt[n]{c} = 1$.

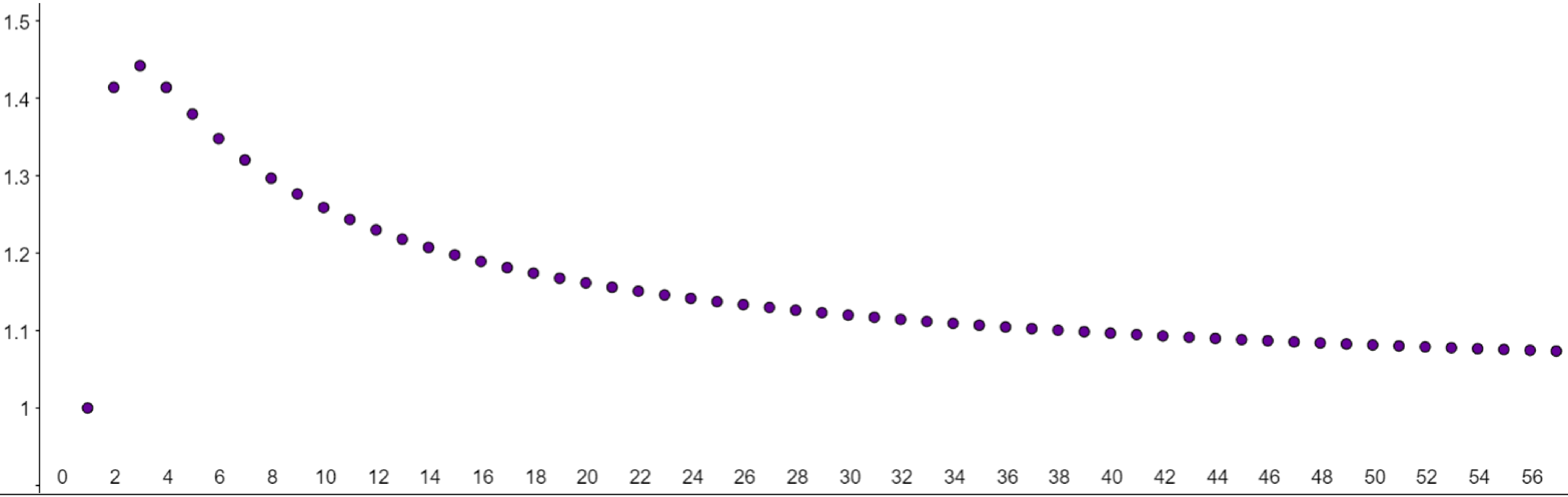
(b) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

$$\lim_{n\to\infty} \sqrt[n]{c} = 1.$$

$c = 3$



$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$



$\mathbf{6}(\boldsymbol{a})$

$$\lim_{n\to\infty} \sqrt[n]{\boldsymbol{n}^3 + \mathbf{3}^{\boldsymbol{n}} + \mathbf{13}^{\boldsymbol{n}} + \ln \boldsymbol{n}} = ?$$

$$\lim_{n\to\infty} \sqrt[n]{\mathbf{13}^{\boldsymbol{n}}\left(\frac{\boldsymbol{n}^3}{\mathbf{13}^{\boldsymbol{n}}} + \frac{\mathbf{3}^{\boldsymbol{n}}}{\mathbf{13}^{\boldsymbol{n}}} + \mathbf{1} + \frac{\ln \boldsymbol{n}}{\mathbf{13}^{\boldsymbol{n}}}\right)} = \lim_{n\to\infty} \mathbf{13} \times \sqrt[n]{\frac{\boldsymbol{n}^3}{\mathbf{13}^{\boldsymbol{n}}} + \frac{\mathbf{3}^{\boldsymbol{n}}}{\mathbf{13}^{\boldsymbol{n}}} + \mathbf{1} + \frac{\ln \boldsymbol{n}}{\mathbf{13}^{\boldsymbol{n}}}} =$$

$$\mathbf{13} \times \mathbf{1} = \mathbf{13}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^3}{13^n} + \frac{3^n}{13^n} + 1 + \frac{\ln n}{13^n}} = ?$$

$$1 < \sqrt[n]{\frac{n^3}{13^n} + \frac{3^n}{13^n} + 1 + \frac{\ln n}{13^n}} < \sqrt[n]{4}$$

$$\left.\begin{array}{l} \textbf{Od jistého indexu: } \dfrac{n^3}{13^n} < 1 \\[2ex] \textbf{Od jistého indexu: } \dfrac{3^n}{13^n} < 1 \\[2ex] \textbf{Od jistého indexu: } \dfrac{\ln n}{13^n} < 1 \end{array}\right\} \quad \frac{n^3}{13^n} + \frac{3^n}{13^n} + 1 + \frac{\ln n}{13^n} < 4$$

$$\mathbf{6}.\,\boldsymbol{b})\ \lim_{n\to\infty}\frac{\sqrt[n]{2n^n+2^n+6^n}}{n}=\lim_{n\to\infty}\frac{\sqrt[n]{n^n\left(2+\frac{2^n}{n^n}+\frac{6^n}{n^n}\right)}}{n}=\lim_{n\to\infty}\frac{\cancel{n}\times\sqrt[n]{2+\frac{2^n}{n^n}+\frac{6^n}{n^n}}}{\cancel{n}}=$$

$$=\lim_{n\to\infty}\sqrt[n]{2+\frac{2^n}{n^n}+\frac{6^n}{n^n}}=\mathbf{1}$$

$$\mathbf{1}<\sqrt[n]{2+\frac{2^n}{n^n}+\frac{6^n}{n^n}}<\sqrt[n]{4}$$

$$\mathbf{6.}\ \boldsymbol{c})\ \lim_{n\to\infty}\frac{\sqrt[2n]{2n^n+2^n+6^n}}{\sqrt{n}+3}=\lim_{n\to\infty}\frac{\sqrt[2n]{n^n\left(2+\frac{2^n}{n^n}+\frac{6^n}{n^n}\right)}}{\sqrt{n}\left(1+\frac{3}{\sqrt{n}}\right)}=\lim_{n\to\infty}\frac{\sqrt{n}\times\sqrt[2n]{2+\frac{2^n}{n^n}+\frac{6^n}{n^n}}}{\sqrt{n}\left(1+\frac{3}{\sqrt{n}}\right)}=$$

$$\lim_{n\to\infty}\frac{\sqrt[2n]{2+\frac{2^n}{n^n}+\frac{6^n}{n^n}}}{1+\frac{3}{\sqrt{n}}}=\frac{\lim_{n\to\infty}\sqrt[2n]{2+\frac{2^n}{n^n}+\frac{6^n}{n^n}}}{\lim_{n\to\infty}\left(1+\frac{3}{\sqrt{n}}\right)}=\frac{1}{1}=1$$

$$1<\sqrt[2n]{2+\frac{2^n}{n^n}+\frac{6^n}{n^n}}<\sqrt[2n]{4}$$

$$\mathbf{6.}\,\boldsymbol{d})\ \lim_{n\to\infty}\frac{\sqrt[n]{3^n+2^n}}{\sqrt[2n]{4^n+3^n}}=\lim_{n\to\infty}\frac{\sqrt[n]{3^n(1+\left(\frac{2}{3}\right)^n)}}{\sqrt[2n]{4^n(1+\left(\frac{3}{4}\right)^n)}}=\lim_{n\to\infty}\frac{3\times\sqrt[n]{1+\left(\frac{2}{3}\right)^n}}{\sqrt{4}\times\sqrt[2n]{1+\left(\frac{3}{4}\right)^n}}=$$

$$\frac{\lim_{n\to\infty}(3\times\sqrt[n]{1+\left(\frac{2}{3}\right)^n})}{\lim_{n\to\infty}(2\times\sqrt[2n]{1+\left(\frac{3}{4}\right)^n})}=\frac{3\times\lim_{n\to\infty}\sqrt[n]{1+\left(\frac{2}{3}\right)^n}}{2\times\lim_{n\to\infty}\sqrt[2n]{1+\left(\frac{3}{4}\right)^n}}=\frac{3\times 1}{2\times 1}=\frac{3}{2}$$

Definice (Eulerovo číslo).

$$\lim \left(1 + \frac{1}{n}\right)^n = e.$$

$$\boxed{x^{n.m} = (x^n)^m}$$

$$\mathbf{7.}\, \boldsymbol{b}) \ \lim_{n\to\infty}\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{n}}\right)^{\boldsymbol{n}} = \lim_{n\to\infty}\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{n}}\right)^{\mathbf{2}\boldsymbol{n}.\frac{\mathbf{1}}{\mathbf{2}}} = \lim_{n\to\infty}\left(\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{n}}\right)^{\mathbf{2}\boldsymbol{n}}\right)^{\frac{\mathbf{1}}{\mathbf{2}}} = \lim_{n\to\infty}\sqrt{\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{n}}\right)^{\mathbf{2}\boldsymbol{n}}}$$

$$= \sqrt{\lim_{n\to\infty}\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{n}}\right)^{\mathbf{2}\boldsymbol{n}}} = \sqrt{\boldsymbol{e}}$$

$\left\{\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{n}}\right)^{\mathbf{2}\boldsymbol{n}}\right\}_{\boldsymbol{n}=\mathbf{1}}^{\infty}$ **je vybranou posloupností z posloupnosti** $\left\{\left(\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{n}}\right)^{\boldsymbol{n}}\right\}_{\boldsymbol{n}=\mathbf{1}}^{\infty}$

$$\mathbf{7.}\ \boldsymbol{c})\ \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} = \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{n} = \lim_{n\to\infty}\left(\left(\frac{n}{n-1}\right)^{-1}\right)^{n} = \lim_{n\to\infty}\left(\left(\frac{n-1+1}{n-1}\right)^{n}\right)^{-1}$$

$$= \left(\lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^{n}\right)^{-1} = \left(\underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^{n-1}}_{\longrightarrow e}\ \underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)}_{\longrightarrow 1}\right)^{-1} = e^{-1}$$