Předmět: NMTM101 Matematická analýza I

Typ výuky: Cvičení

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Řady

$$\sum_{n=1}^{\infty} a_n$$

Geometrická řada

$$\sum_{n=0}^{\infty} q^n$$

$$= \begin{cases} \dfrac{1}{1-q} & -1 < q < 1 \\ \text{Divergentní} & q \geq 1 \vee q \leq -1 \end{cases}$$

$$\sum_{n=1}^{\infty} 2^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

geometrická řada

$$\frac{1}{2} < 1$$

$$\sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^{\frac{n}{2}} = \sum_{n=1}^{\infty} \left(\sqrt{\frac{5}{6}}\right)^{n} = \frac{\sqrt{\frac{5}{6}}}{1-\sqrt{\frac{5}{6}}}$$

geometrická řada

$$\sqrt{\frac{5}{6}} < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \begin{cases} \textbf{Konvergentní} & \alpha > 1 \\ \textbf{Divergentní} & \alpha \leq 1 \end{cases}$$

$$\sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$$ **diverguje**

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \begin{cases} \text{Konvergentní} & \alpha > 1 \\ \text{Divergentní} & \alpha \leq 1 \end{cases}$$

$$\sum_{n=1}^{\infty} n^{-\frac{3}{2}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \qquad \frac{3}{2} > 1$$ **Konvergentní**

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} \qquad \frac{1}{2} < 1$$ **Divergentní**

Věta (Nutná podmínka konvergence řady).

Nechť $\sum_{n=1}^{\infty} a_n$ konverguje, pak $\lim_{n\to\infty} a_n = 0$

$$\sum_{n=1}^{\infty} a_n \textbf{ konverguje} \Rightarrow \lim_{n\to\infty} a_n = 0$$

$$\lim_{n\to\infty} a_n \neq 0 \quad \textbf{nebo } \lim_{n\to\infty} a_n \textbf{ neexistuje} \quad \Rightarrow \sum_{n=1}^{\infty} a_n \textbf{ diverguje}$$

$$\sum_{n=1}^{\infty} \frac{n+1}{2n+4} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{2n+4} = \frac{1}{2} \neq 0 \qquad \sum_{n=1}^{\infty} \frac{n+1}{2n+4} \text{ diverguje}$$

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0 \qquad \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n \text{ diverguje}$$

Věta (Srovnávací kriterium).

Nechť $\sum_{n=1}^{\infty} a_n$ a $\sum_{n=1}^{\infty} b_n$ jsou řady s nezápornými členy, které splňují $\forall n \in \mathbb{N} \ : a_n \leqslant b_n$, pak platí:

(a) $\sum_{n=1}^{\infty} b_n$ konverguje $\implies \sum_{n=1}^{\infty} a_n$ konverguje

(b) $\sum_{n=1}^{\infty} a_n$ diverguje $\implies \sum_{n=1}^{\infty} b_n$ diverguje

Vyšetřete konvergenci následujících řad:

$$\sum_{n=1}^{\infty} \frac{17n+2}{n^2} \quad \to \frac{1}{n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{17n+2}{n^2} = 0$$

$$\frac{1}{n} \qquad \frac{17n+2}{n^2}$$

$$n < 17n+2 \quad \Rightarrow \frac{n}{n^2} < \frac{17n+2}{n^2} \quad \Rightarrow \frac{1}{n} < \frac{17n+2}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ **diverguje** $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{17n+2}{n^2}$ **diverguje**

$$\sum_{n=1}^{\infty} \frac{\log n}{n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\log n}{n} = 0$$

$$1 < \log n \iff \frac{1}{n} < \frac{\log n}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverguje } \Rightarrow \sum_{n=1}^{\infty} \frac{\log n}{n} \text{ diverguje}$$

$$\sum_{n=1}^{\infty} \frac{\log n}{n^2} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\log n}{n^2} = 0$$

Věta (Růstová Škála)

$$\frac{\log n}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ **konverguje** $\Rightarrow \sum_{n=1}^{\infty} \frac{\log n}{n^2}$ **konverguje**

$$\sum_{n=1}^{\infty} \frac{1}{n13^n} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n13^n} = 0$$

$$\left(\frac{1}{13}\right)^n$$

$$\frac{1}{n13^n} < \frac{1}{13^n} = \left(\frac{1}{13}\right)^n$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{13}\right)^n \text{ konverguje} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n13^n} \text{ konverguje}$$

Věta (Limitní srovnávací kriterium). Nechť $\sum_{n=1}^{\infty} a_n$ a $\sum_{n=1}^{\infty} b_n$ jsou řady s nezápornými členy a nechť existuje limita:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = A \in \mathbb{R}^*$$

(a) Pro $A \in (0, \infty)$ platí: $\sum_{n=1}^{\infty} b_n$ konverguje $\iff$ $\sum_{n=1}^{\infty} a_n$ konverguje

(b) Pro $A \in (0, \infty)$ platí: $\sum_{n=1}^{\infty} a_n$ diverguje $\iff$ $\sum_{n=1}^{\infty} b_n$ diverguje

(c) Pro $A = 0$ platí: $\sum_{n=1}^{\infty} b_n$ konverguje $\implies$ $\sum_{n=1}^{\infty} a_n$ konverguje

(d) Pro $A = \infty$ platí: $\sum_{n=1}^{\infty} b_n$ diverguje $\implies$ $\sum_{n=1}^{\infty} a_n$ diverguje

Vyšetřete konvergenci následujících řad:

$$\sum_{n=1}^{\infty} \frac{17}{n^2} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{17}{n^2} = 0$$

$$\sum_{n=1}^{\infty} \frac{17}{n^2} = 17 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

konverguje

konverguje

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ konverguje}$$

$$b_n = \frac{1}{n^2}$$

$$a_n = \frac{17}{n^2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{17}{n^2}}{\frac{1}{n^2}} = 17 \quad \overset{LSK}{\Rightarrow} \quad \sum_{n=1}^{\infty} \frac{17}{n^2} \text{ konverguje}$$

$$\sum_{n=1}^{\infty} \sin\frac{1}{n} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\frac{1}{n} = 0 \qquad a_n = \sin\frac{1}{n} \sim \frac{1}{n} \qquad b_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$$

$\sum_{n=1}^{\infty} b_n$ **diverguje** $\Rightarrow \sum_{n=1}^{\infty} a_n$ **diverguje**

$$\sum_{n=1}^{\infty} \frac{\log(1+\frac{1}{n})}{n^{\frac{5}{2}}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\log(1+\frac{1}{n})}{n^{\frac{5}{2}}} = 0$$

$$a_n = \frac{\log(1+\frac{1}{n})}{n^{\frac{5}{2}}} \quad \log(1+\frac{1}{n}) \sim \frac{1}{n}$$

$$b_n = \frac{\frac{1}{n}}{n^{\frac{5}{2}}} = \frac{1}{n.n^{\frac{5}{2}}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\log(1+\frac{1}{n})}{n^{\frac{5}{2}}}}{\frac{1}{n.n^{\frac{5}{2}}}} = \lim_{n\to\infty} \frac{\log(1+\frac{1}{n})}{\frac{1}{n}} = 1$$

$\sum_{n=1}^{\infty} b_n$ **konverguje** $\Rightarrow \sum_{n=1}^{\infty} a_n$ **konverguje**

$$\sum_{n=1}^{\infty} \frac{1+n^2}{1+n^4} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1+n^2}{1+n^4} = 0$$

$$\frac{n^2}{n^4} = \frac{1}{n^2}$$

$$a_n = \frac{1+n^2}{1+n^4} \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1+n^2}{1+n^4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2+n^4}{1+n^4} = 1$$

$$b_n = \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ konverguje } \overset{LSK}{\Rightarrow} \sum_{n=1}^{\infty} \frac{1+n^2}{1+n^4} \text{ konverguje}$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)\frac{1+n^2}{1+n^3} \qquad a_n = \sin\left(\frac{1}{n}\right)\frac{1+n^2}{1+n^3}$$

$$\sim \frac{1}{n} \qquad \sim \frac{n^2}{n^3} = \frac{1}{n} \qquad b_n = \frac{1}{n}\frac{1}{n}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin\left(\frac{1}{n}\right)\frac{1+n^2}{1+n^3}}{\frac{1}{n}\frac{1}{n}} = \lim_{n\to\infty}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\lim_{n\to\infty}\frac{\frac{1+n^2}{1+n^3}}{\frac{1}{n}}$$

$$= 1 \lim_{n\to\infty}\frac{n+n^3}{1+n^3} = 1$$

$\sum_{n=1}^{\infty} b_n$ **konverguje** $\Rightarrow \sum_{n=1}^{\infty} a_n$ **konverguje**

$$\sum_{n=1}^{\infty}\left(\sqrt[n]{2}-1\right) arctg\, 2n \qquad a_n = \left(\sqrt[n]{2}-1\right) arctg\, 2n = \left(2^{\frac{1}{n}}-1\right) arctg\, 2n$$

$$= \left(e^{\frac{1}{n}\ln 2}-1\right) arctg\, 2n$$

$$\sim \frac{1}{n}\ln 2 \qquad \to \frac{\pi}{2} \qquad b_n = \frac{1}{n}\ln 2$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\left(e^{\frac{1}{n}\ln 2}-1\right) arctg\, 2n}{\frac{1}{n}\ln 2} = \frac{\pi}{2}$$

$$\to 1$$

$\sum_{n=1}^{\infty} b_n$ **diverguje** $\Rightarrow \sum_{n=1}^{\infty} a_n$ **diverguje**

Věta (Cauchyho limitní odmocninové kritérium).

Nechť $\sum_{n=1}^{\infty} a_n$ je řada s nezápornými členy. Pak platí:

(a) $\lim_{n\to\infty} \sqrt[n]{a_n} < 1 \implies \sum_{n=1}^{\infty} a_n$ Konverguje

(b) $\lim_{n\to\infty} \sqrt[n]{a_n} > 1 \implies \sum_{n=1}^{\infty} a_n$ Diverguje

(c) $\lim_{n\to\infty} \sqrt[n]{a_n} = 1 \implies \sum_{n=1}^{\infty} a_n$ Řada může konvergovat i divergovat.

Věta (D'Alembertovo limitní podílové kritérium).

Nechť $\sum_{n=1}^{\infty} a_n$ je řada s nezápornými členy.

Pak platí:

(a) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \implies \sum_{n=1}^{\infty} a_n$ Konverguje

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(c) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \implies \sum_{n=1}^{\infty} a_n$ Řada může konvergovat i divergovat.

$$\sum_{n=1}^{\infty} \frac{1}{1+4^n} \qquad a_n = \frac{1}{1+4^n} \quad a_{n+1} = \frac{1}{1+4^{n+1}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{1+4^{n+1}}}{\frac{1}{1+4^n}} = \lim_{n\to\infty} \frac{1+4^n}{1+4^{n+1}} = \lim_{n\to\infty} \frac{4^n(\frac{1}{4^n}+1)}{4^{n+1}(\frac{1}{4^{n+1}}+1)}$$

$$= \frac{1}{4} \quad < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{1+4^n} \ \textbf{konverguje}$$

$$\sum_{n=1}^{\infty} \frac{1}{1+4^n} \qquad a_n = \frac{1}{1+4^n}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{1+4^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{1}}{\sqrt[n]{1+4^n}} = \frac{\lim\limits_{n\to\infty} \sqrt[n]{1}}{\lim\limits_{n\to\infty} \sqrt[n]{1+4^n}} = \frac{1}{4} < 1$$

$$4 = \sqrt[n]{4^n} < \sqrt[n]{1+4^n} < \sqrt[n]{4^n+4^n} = \sqrt[n]{2.4^n} = 4\sqrt[n]{2}$$

$$\rightarrow 4$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{1+4^n} = 4$$

$$\sum_{n=1}^{\infty} \frac{1}{1+4^n} \ \textbf{konverguje}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{1+3^n} \qquad a_n = \frac{n^2}{1+3^n}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^2}{1+3^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{\sqrt[n]{1+3^n}} = \frac{\lim\limits_{n\to\infty} \sqrt[n]{n^2}}{\lim\limits_{n\to\infty} \sqrt[n]{1+3^n}} = \frac{\lim\limits_{n\to\infty} \sqrt[n]{n} \ \lim\limits_{n\to\infty} \sqrt[n]{n}}{3} = \frac{1}{3} < 1$$

$$\sum_{n=1}^{\infty} \frac{n^2}{1+3^n} \ \textbf{konverguje}$$

$$3 = \sqrt[n]{3^n} < \sqrt[n]{1+3^n} < \sqrt[n]{3^n+3^n} = \sqrt[n]{2.3^n} = 3\sqrt[n]{2}$$

$$\to 3$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{1+3^n} = 3$$

$$\sum_{n=1}^{\infty} \frac{n^2}{1+3^n} \qquad a_n = \frac{n^2}{1+3^n} \quad a_{n+1} = \frac{(n+1)^2}{1+3^{n+1}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{1+3^{n+1}}}{\frac{n^2}{1+3^n}} = \lim_{n\to\infty} \frac{(n+1)^2(1+3^n)}{n^2(1+3^{n+1})} = \lim_{n\to\infty} \frac{n^2 3^n + \cdots}{n^2 3^{n+1} + \cdots} = \cdots = \frac{1}{3} < 1$$

$$\sum_{n=1}^{\infty} \frac{n^2}{1+3^n} \text{ konverguje}$$

$$\sum_{n=1}^{\infty} \frac{e^n\sqrt{n}}{3^n} \qquad a_n = \frac{e^n\sqrt{n}}{3^n}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{e^n\sqrt{n}}{3^n}} = \lim_{n\to\infty} \frac{e}{3} \sqrt[n]{\sqrt{n}} = \frac{e}{3} \lim_{n\to\infty} \sqrt{\sqrt[n]{n}} = \frac{e}{3} \sqrt{\lim_{n\to\infty} \sqrt[n]{n}}$$

$$= \frac{e}{3} < 1$$

$$\sum_{n=1}^{\infty} \frac{e^n\sqrt{n}}{3^n} \quad \textbf{konverguje}$$

$$\sum_{n=1}^{\infty} \frac{c^n}{n!} \qquad a_n = \frac{c^n}{n!}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{c^{n+1}}{(n+1)!}}{\frac{c^n}{n!}} = \lim_{n\to\infty} \frac{n!\, c^{n+1}}{(n+1)!\, c^n} = \lim_{n\to\infty} \frac{\cancel{n!}\, \cancel{c^n} c}{(n+1)\cancel{n!}\, \cancel{c^n}}$$

$$= 0 < 1$$

$$\sum_{n=1}^{\infty} \frac{c^n}{n!} \quad \textbf{konverguje}$$

$$\sum_{n=1}^{\infty} \sqrt[3]{n^2+5} - \sqrt[3]{n^2+1} \qquad \sqrt[3]{n^2+5} = A \qquad \sqrt[3]{n^2+1} = B$$

$$\sqrt[3]{n^2+5} - \sqrt[3]{n^2+1} \, \frac{(A^2+AB+B^2)}{(A^2+AB+B^2)} = \frac{n^2+5-n^2-1}{(\sqrt[3]{n^2+5})^2+(\sqrt[3]{n^2+5})(\sqrt[3]{n^2+1})+(\sqrt[3]{n^2+1})^2}$$

$n^2+5-n^2-1 \rightarrow 4$

$$\sum_{n=1}^{\infty} \frac{4}{(\sqrt[3]{n^2+5})^2+(\sqrt[3]{n^2+5})(\sqrt[3]{n^2+1})+(\sqrt[3]{n^2+1})^2}$$

$$a_n = \frac{4}{(\sqrt[3]{n^2+5})^2+(\sqrt[3]{n^2+5})(\sqrt[3]{n^2+1})+(\sqrt[3]{n^2+1})^2} \sim \frac{4}{3n^{\frac{4}{3}}} \qquad b_n = \frac{1}{n^{\frac{4}{3}}}$$

$\sim n^{\frac{4}{3}}$ $\quad \sim n^{\frac{4}{3}}$ $\quad \sim n^{\frac{4}{3}}$

$$\sum_{n=1}^{\infty} \sqrt[3]{n^2+5} - \sqrt[3]{n^2+1}$$

$$= \sum_{n=1}^{\infty} \frac{4}{(\sqrt[3]{n^2+5})^2 + (\sqrt[3]{n^2+5})(\sqrt[3]{n^2+1}) + (\sqrt[3]{n^2+1})^2}$$

$$a_n = \frac{4}{(\sqrt[3]{n^2+5})^2 + (\sqrt[3]{n^2+5})(\sqrt[3]{n^2+1}) + (\sqrt[3]{n^2+1})^2} \qquad b_n = \frac{1}{n^{\frac{4}{3}}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{4}{(n^2+5)^{\frac{2}{3}} + (n^2+5)^{\frac{1}{3}}(n^2+1)^{\frac{1}{3}} + (n^2+1)^{\frac{2}{3}}}}{\dfrac{1}{n^{\frac{4}{3}}}}$$

$$n^{\frac{4}{3}}(1+\frac{5}{n^2})^{\frac{2}{3}} + n^{\frac{2}{3}}(1+\frac{5}{n^2})^{\frac{1}{3}} n^{\frac{2}{3}}(1+\frac{1}{n^2})^{\frac{1}{3}} + n^{\frac{4}{3}}(1+\frac{1}{n^2})^{\frac{2}{3}}$$

$$= \lim_{n\to\infty} \frac{4n^{\frac{4}{3}}}{n^{\frac{4}{3}}[(1+\frac{5}{n^2})^{\frac{2}{3}} + (1+\frac{5}{n^2})^{\frac{1}{3}} (1+\frac{1}{n^2})^{\frac{1}{3}} + (1+\frac{1}{n^2})^{\frac{2}{3}}]} = \frac{4}{3}$$

$$\sum_{n=1}^{\infty} b_n \text{ K} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ K}$$

Věta (Cauchyho limitní odmocninové kritérium).

Nechť $\sum_{n=1}^{\infty} a_n$ je řada s nezápornými členy. Pak platí:

(a) $\lim_{n\to\infty} \sqrt[n]{a_n} < 1 \implies \sum_{n=1}^{\infty} a_n$ Konverguje

(b) $\lim_{n\to\infty} \sqrt[n]{a_n} > 1 \implies \sum_{n=1}^{\infty} a_n$ Diverguje

(c) $\lim_{n\to\infty} \sqrt[n]{a_n} = 1 \implies \sum_{n=1}^{\infty} a_n$ Řada může konvergovat i divergovat.

Věta (D'Alembertovo limitní podílové kritérium).

Nechť $\sum_{n=1}^{\infty} a_n$ je řada s nezápornými členy.

Pak platí:

(a) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \implies \sum_{n=1}^{\infty} a_n$ Konverguje

(b) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1 \implies \sum_{n=1}^{\infty} a_n$ Diverguje

(c) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \implies \sum_{n=1}^{\infty} a_n$ Řada může konvergovat i divergovat.

$$\sum_{n=1}^{\infty} \frac{n^2}{1+3^n} \qquad a_n = \frac{n^2}{1+3^n}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^2}{1+3^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{\sqrt[n]{1+3^n}} = \frac{\lim_{n\to\infty} \sqrt[n]{n^2}}{\lim_{n\to\infty} \sqrt[n]{1+3^n}} = \frac{\lim_{n\to\infty} \sqrt[n]{n} \ \lim_{n\to\infty} \sqrt[n]{n}}{3} = \frac{1}{3} < 1$$

LOK

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{1+3^n} \ \text{konverguje}$$

$$3 = \sqrt[n]{3^n} < \sqrt[n]{1+3^n} < \sqrt[n]{3^n+3^n} = \sqrt[n]{2.\,3^n} = 3\sqrt[n]{2}$$

$$\to 3$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{1+3^n} = 3$$

$$\sum_{n=1}^{\infty} \frac{n^2}{1+3^n} \qquad a_n = \frac{n^2}{1+3^n} \quad a_{n+1} = \frac{(n+1)^2}{1+3^{n+1}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{1+3^{n+1}}}{\frac{n^2}{1+3^n}} = \lim_{n\to\infty} \frac{(n+1)^2(1+3^n)}{n^2(1+3^{n+1})} = \lim_{n\to\infty} \frac{n^2 3^n + \cdots}{n^2 3^{n+1} + \cdots} = \cdots = \frac{1}{3} < 1$$

LPK

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{1+3^n} \quad \textbf{konverguje}$$

$$\sum_{n=1}^{\infty} \frac{c^n}{n!} \qquad a_n = \frac{c^n}{n!}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{c^{n+1}}{(n+1)!}}{\frac{c^n}{n!}} = \lim_{n\to\infty} \frac{n!\, c^{n+1}}{(n+1)!\, c^n} = \lim_{n\to\infty} \frac{n!\, c^n c}{(n+1) n!\, c^n} = \lim_{n\to\infty} \frac{c}{(n+1)} = 0 < 1$$

LPK

$$\Rightarrow \sum_{n=1}^{\infty} \frac{c^n}{n!} \quad \textbf{konverguje}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{2^{n^2}} \qquad a_n = \frac{(n!)^2}{2^{n^2}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{2^{(n+1)^2}}}{\frac{(n!)^2}{2^{n^2}}} = \lim_{n\to\infty} \frac{((n+1)!)^2 2^{n^2}}{(n!)^2 2^{(n+1)^2}} = \lim_{n\to\infty} \frac{((n+1)n!)^2\, 2^{n^2}}{(n!)^2\, 2^{n^2+2n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 \cancel{(n!)^2}\, \cancel{2^{n^2}}}{\cancel{(n!)^2}\, \cancel{2^{n^2}}\, 2^{2n+1}} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{2n+1}} = 0 < 1$$

LPK

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(n!)^2}{2^{n^2}} \quad \textbf{konverguje}$$

$$\sum_{n=1}^{\infty} \frac{n^5 + 9^n}{\sqrt{n} + e^{2n}} \qquad a_n = \frac{n^5 + 9^n}{\sqrt{n} + e^{2n}}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^5 + 9^n}{\sqrt{n} + e^{2n}}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^5 + 9^n}}{\sqrt[n]{\sqrt{n} + e^{2n}}} = \frac{\lim\limits_{n\to\infty} \sqrt[n]{n^5 + 9^n}}{\lim\limits_{n\to\infty} \sqrt[n]{\sqrt{n} + e^{2n}}} = \frac{9}{e^2} > 1$$

LOK

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n^5 + 9^n}{\sqrt{n} + e^{2n}} \quad \textbf{diverguje}$$

$$9 = \sqrt[n]{9^n} < \sqrt[n]{n^5 + 9^n} < \sqrt[n]{9^n + 9^n} = \sqrt[n]{2.9^n} = 9\sqrt[n]{2} \Rightarrow \lim_{n\to\infty} \sqrt[n]{n^5 + 9^n} = 9$$

$$\to 9$$

$$e^2 = \sqrt[n]{e^{2n}} < \sqrt[n]{\sqrt{n} + e^{2n}} < \sqrt[n]{e^{2n} + e^{2n}} = \sqrt[n]{2.e^{2n}} = e^2\sqrt[n]{2} \Rightarrow \lim_{n\to\infty} \sqrt[n]{\sqrt{n} + e^{2n}} = e^2$$

$$\to e^2$$

$$\sum_{n=1}^{\infty} \frac{2.5.8..(3n-1)}{3.8.13..(5n-2)} \qquad a_n = \frac{2.5.8..(3n-1)}{3.8.13..(5n-2)} \quad a_{n+1} = \frac{2.5.8..(3n-1)(3n+2)}{3.8.13..(5n-2)(5n+3)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{2.5.8..(3n-1)(3n+2)}{3.8.13..(5n-2)(5n+3)}}{\dfrac{2.5.8..(3n-1)}{3.8.13..(5n-2)}} = \lim_{n\to\infty} \frac{3n+2}{5n+3} = \frac{3}{5} < 1$$

LPK

$$\Rightarrow \sum_{n=1}^{\infty} \frac{2.5.8..(3n-1)}{3.8.13..(5n-2)} \quad \textbf{konverguje}$$

Absolutní konvergence:

Řekneme, že řada $\sum_{n=1}^{\infty} a_n$ je absolutně konvergentní, konverguje-li řada $\sum_{n=1}^{\infty} |a_n|$.

$$\sum_{n=1}^{\infty} a_n \text{ je absolutně konvergentní} \Leftrightarrow \sum_{n=1}^{\infty} |a_n| \text{ konverguje}$$

Věta: **Každá absolutně konvergentní řada je konvergentní.**

$$\sum_{n=1}^{\infty} |a_n| \text{ konverguje} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ konverguje}$$

Relativní konvergence:

Řekneme, že řada $\sum_{n=1}^{\infty} a_n$ je relativně konvergentní (též: neabsolutně konvergentní), jestliže je konvergentní, ale není absolutně konvergentní.

$$\sum_{n=1}^{\infty} a_n \text{ je relativně konvergentní} \Leftrightarrow \sum_{n=1}^{\infty} |a_n| \text{ diverguje} \vee \sum_{n=1}^{\infty} a_n \text{ konverguje}$$

$$\sum_{n=1}^{\infty}(-1)^n a_n$$

Věta (Leibnizovo kritérium): **Budiž** $\{a_n\}_{n=1}^{\infty}$

- **nerostoucí posloupnost reálných čísel**
- **s** $\lim_{n\to\infty} a_n = 0.$

Potom řada $\sum_{n=1}^{\infty}(-1)^n a_n$ **konverguje.**

Vyšetřete absolutní a relativní konvergenci následujících řad:

$$\sum_{n=1}^{\infty}(-1)^n(\sqrt[n]{5}-1)$$

$$\sum_{n=1}^{\infty}|(-1)^n(\sqrt[n]{5}-1)| = \sum_{n=1}^{\infty}|\underbrace{\sqrt[n]{5}-1}_{>0}| = \sum_{n=1}^{\infty}(\sqrt[n]{5}-1) = \sum_{n=1}^{\infty}(e^{\frac{1}{n}\ln 5}-1)$$

$$a_n = e^{\frac{1}{n}\ln 5} - 1 \sim \frac{1}{n}\ln 5 = b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{e^{\frac{1}{n}\ln 5}-1}{\frac{1}{n}\ln 5} = 1$$

LSK

$\sum_{n=1}^{\infty}(\frac{1}{n}\ln 5)$ **diverguje** $\Rightarrow \sum_{n=1}^{\infty}(e^{\frac{1}{n}\ln 5}-1)$ **diverguje**

$\sum_{n=1}^{\infty}|(-1)^n(\sqrt[n]{5}-1)|$ **diverguje** $\Rightarrow \sum_{n=1}^{\infty}(-1)^n(\sqrt[n]{5}-1)$ **není absolutně konvergentní.**

$$\sum_{n=1}^{\infty}(-1)^n\left(\sqrt[n]{5}-1\right)$$ **není absolutně konvergentní.**

$$\sum_{n=1}^{\infty}(-1)^n(\sqrt[n]{5}-1)=\sum_{n=1}^{\infty}(-1)^n a_n \qquad a_n=\sqrt[n]{5}-1$$

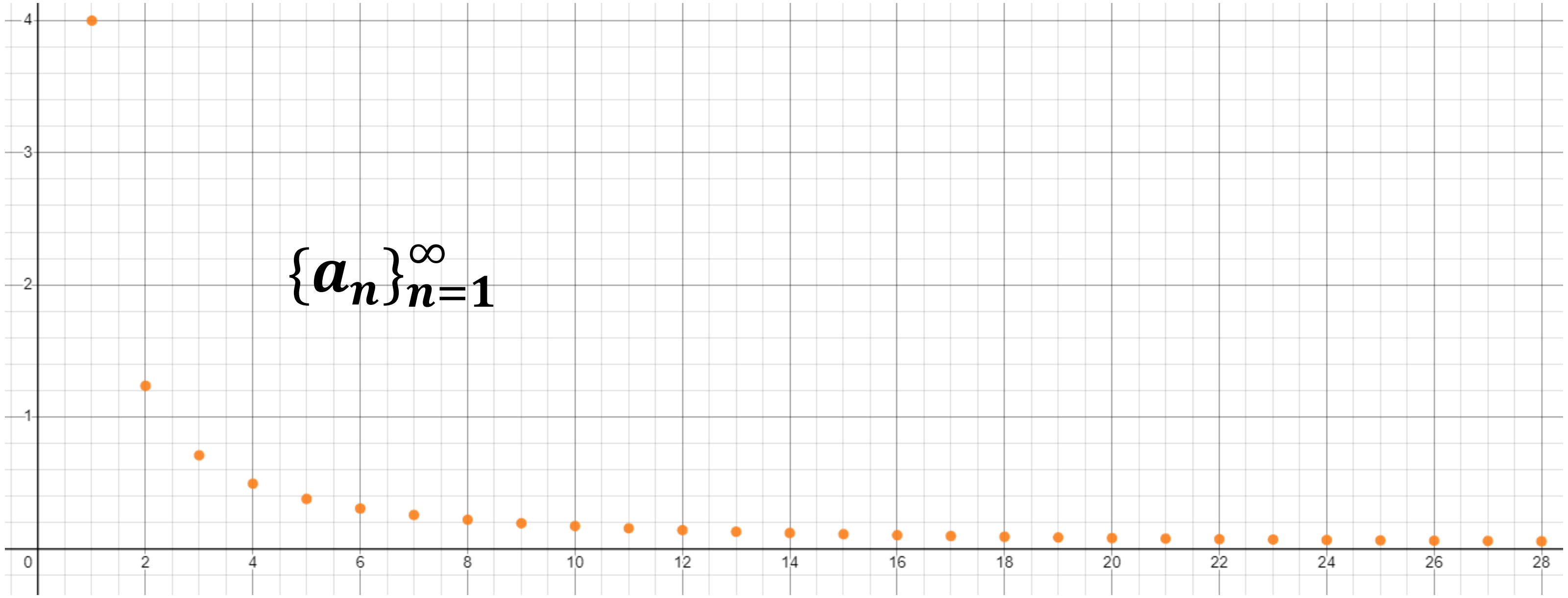
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(\sqrt[n]{5}-1) = 0$$

$$a_n=\sqrt[n]{5}-1 \qquad a_{n+1}=\sqrt[n+1]{5}-1$$

$$\sqrt[n+1]{5} < \sqrt[n]{5}$$

$$\sqrt[n+1]{5}-1 < \sqrt[n]{5}-1$$

$$a_{n+1} < a_n$$ **klesající**



Leibnizovo kritérium

$$\Rightarrow \sum_{n=1}^{\infty}(-1)^n(\sqrt[n]{5}-1)$$ **konverguje**

$\sum_{n=1}^{\infty}(-1)^n(\sqrt[n]{5}-1)$ **je relativně konvergentní (též: neabsolutně konvergentní)**

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2+3n-1}{4n^2-n-6}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n \frac{n^2+3n-1}{4n^2-n-6}$$ **neexistuje**

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2+3n-1}{4n^2-n-6}$$ **diverguje**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{1}{n}$$

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n} \sin\frac{1}{n}\right| = \sum_{n=1}^{\infty} |(-1)^n| \left|\underbrace{\frac{1}{n} \sin\frac{1}{n}}_{>0}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{1}{n}$$

$$a_n = \frac{1}{n} \sin\frac{1}{n} \sim \frac{1}{n} \qquad b_n = \frac{1}{n}\frac{1}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n} \sin\frac{1}{n}}{\frac{1}{n}\frac{1}{n}} = 1$$

LSK

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ konverguje $\Longrightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{1}{n}$ konverguje

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n} \sin\frac{1}{n}\right|$ konverguje

Řada $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{1}{n}$ je absolutně konvergentní

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\log\log n}$$

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\log\log n} \right| = \sum_{n=2}^{\infty} \frac{|(-1)^n|}{|\log\log n|} = \sum_{n=2}^{\infty} \frac{1}{|\log\log n|} = \frac{1}{|\log\log 2|} + \sum_{n=3}^{\infty} \frac{1}{|\log\log n|}$$

$$= -\frac{1}{\log\log 2} + \sum_{n=3}^{\infty} \frac{1}{\log\log n}$$

$$\log\log n < n \quad \Rightarrow \frac{1}{n} < \frac{1}{\log\log n}$$

$$SK$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ **diverguje** $\Longrightarrow \sum_{n=3}^{\infty} \frac{1}{log\ log\ n}$ **diverguje**

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\log\log n} \right|$ **diverguje** $\Longrightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{\log\log n}$ **není absolutně konvergentní**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\log \log n}$$ **není absolutně konvergentní**

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\log \log n} = \sum_{n=1}^{\infty} (-1)^n a_n \qquad a_n = \frac{1}{\log \log n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\log \log n} = 0 \qquad a_{n+1} = \frac{1}{\log \log (n+1)} < \frac{1}{\log \log n} = a_n$$ **klesající**

Leibnizovo kritérium

$$\Rightarrow \sum_{n=2}^{\infty} \frac{(-1)^n}{\log \log n}$$ **konverguje**

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\log \log n}$ **je relativně konvergentní (též: neabsolutně konvergentní)**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+(-1)^n}$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2n+(-1)^n} \right| = \sum_{n=1}^{\infty} \frac{|(-1)^n|}{|\underbrace{2n+(-1)^n}_{>0}|} = \sum_{n=1}^{\infty} \frac{1}{2n+(-1)^n}$$

$$a_n = \frac{1}{2n+(-1)^n} \qquad b_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2n+(-1)^n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+(-1)^n} = \frac{1}{2} > 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverguje} \overset{LSK}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{1}{2n+(-1)^n} \text{ diverguje}$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2n+(-1)^n} \right| \text{ diverguje} \Longrightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+(-1)^n} \text{ není absolutně konvergentní}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+(-1)^n} = \sum_{n=1}^{\infty}(-1)^n a_n \qquad a_n = \frac{1}{2n+(-1)^n}$$

$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+(-1)^n}$ **není absolutně konvergentní**

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{2n+(-1)^n} = 0$$

n je liché: $\quad a_{n+1} = \frac{1}{2n+3} < a_n = \frac{1}{2n-1}$

n je sudé: $\quad \frac{1}{2n+5} = a_{n+2} < a_{n+1} = \frac{1}{2n+1} = a_n = \frac{1}{2n+1}$

n	1	2	3	4	5	$\cdots$
$a_n = \frac{1}{2n+(-1)^n}$	1	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{9}$	$\frac{1}{9}$	$\cdots$

je nerostoucí

Leibnizovo kritérium

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{2n+(-1)^n}$$

konverguje

$\sum_{n=2}^{\infty}\frac{(-1)^n}{2n+(-1)^n}$ **je relativně konvergentní**

$$\sum_{n=1}^{\infty} \frac{\log(2^{2^n}+1)}{\log(2^{4^n}+1)} = \sum_{n=1}^{\infty} \frac{\log(2^{2^n}(1+\frac{1}{2^{2^n}}))}{\log(2^{4^n}(1+\frac{1}{2^{4^n}}))} = \sum_{n=1}^{\infty} \frac{\log(2^{2^n})+\log(1+\frac{1}{2^{2^n}})}{\log(2^{4^n})+\log(1+\frac{1}{2^{4^n}})}$$

$$= \sum_{n=1}^{\infty} \frac{2^n\log(2)+\log(1+\frac{1}{2^{2^n}})}{4^n\log(2)+\log(1+\frac{1}{2^{4^n}})} \qquad a_n = \frac{2^n\log(2)+\log(1+\frac{1}{2^{2^n}})}{4^n\log(2)+\log(1+\frac{1}{2^{4^n}})}$$

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\frac{2^n\log(2)+\log(1+\frac{1}{2^{2^n}})}{4^n\log(2)+\log(1+\frac{1}{2^{4^n}})}} = \lim_{n\to\infty}\frac{\sqrt[n]{2^n\log(2)+\log(1+\frac{1}{2^{2^n}})}}{\sqrt[n]{4^n\log(2)+\log(1+\frac{1}{2^{4^n}})}}$$

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\frac{2^n\log(2)+\log(1+\frac{1}{2^{2^n}})}{4^n\log(2)+\log(1+\frac{1}{2^{4^n}})}} = \lim_{n\to\infty}\frac{\sqrt[n]{2^n\log(2)+\log(1+\frac{1}{2^{2^n}})}}{\sqrt[n]{4^n\log(2)+\log(1+\frac{1}{2^{4^n}})}} = \frac{2}{4} = \frac{1}{2} < 1$$

Konverguje

$$2\sqrt[n]{\log(2)} = \sqrt[n]{2^n\log(2)} < \sqrt[n]{2^n\log(2)+\log(1+\frac{1}{2^{2^n}})} < \sqrt[n]{2^n\log(2)+2^n\log(2)}$$

$$= \sqrt[n]{2.2^n\log(2)} = 2\sqrt[n]{2\log(2)}$$

$\to 2$ $\quad$ $\to 2$ $\quad$ $\to 2$

$$4\sqrt[n]{\log(2)} = \sqrt[n]{4^n\log(2)} < \sqrt[n]{4^n\log(2)+\log(1+\frac{1}{2^{4^n}})} < \sqrt[n]{4^n\log(2)+4^n\log(2)}$$

$$= \sqrt[n]{2.4^n\log(2)} = 4\sqrt[n]{2\log(2)}$$

$\to 4$ $\quad$ $\to 4$ $\quad$ $\to 4$

$$\sum_{n=1}^{\infty} \sqrt[3]{n^2+5} - \sqrt[3]{n^2+1}$$

$$\sqrt[3]{n^2+5} = A \qquad \sqrt[3]{n^2+1} = B$$

$$\sqrt[3]{n^2+5} - \sqrt[3]{n^2+1}\ \frac{(A^2+AB+B^2)}{(A^2+AB+B^2)} = \frac{n^2+5-n^2-1}{(\sqrt[3]{n^2+5})^2+(\sqrt[3]{n^2+5})(\sqrt[3]{n^2+1})+(\sqrt[3]{n^2+1})^2}$$

$n^2+5-n^2-1 \to 4$

$$\sum_{n=1}^{\infty} \frac{4}{(\sqrt[3]{n^2+5})^2+(\sqrt[3]{n^2+5})(\sqrt[3]{n^2+1})+(\sqrt[3]{n^2+1})^2}$$

$$a_n = \frac{4}{(\sqrt[3]{n^2+5})^2+(\sqrt[3]{n^2+5})(\sqrt[3]{n^2+1})+(\sqrt[3]{n^2+1})^2} \sim \frac{4}{3n^{\frac{4}{3}}} \qquad b_n = \frac{1}{n^{\frac{4}{3}}}$$

$\sim n^{\frac{4}{3}}$ $\qquad \sim n^{\frac{4}{3}}$ $\qquad \sim n^{\frac{4}{3}}$

$$\sum_{n=1}^{\infty} \sqrt[3]{n^2+5} - \sqrt[3]{n^2+1}$$

$$= \sum_{n=1}^{\infty} \frac{4}{(\sqrt[3]{n^2+5})^2 + (\sqrt[3]{n^2+5})(\sqrt[3]{n^2+1}) + (\sqrt[3]{n^2+1})^2}$$

$$a_n = \frac{4}{(\sqrt[3]{n^2+5})^2 + (\sqrt[3]{n^2+5})(\sqrt[3]{n^2+1}) + (\sqrt[3]{n^2+1})^2} \qquad b_n = \frac{1}{n^{\frac{4}{3}}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{4}{(n^2+5)^{\frac{2}{3}} + (n^2+5)^{\frac{1}{3}}(n^2+1)^{\frac{1}{3}} + (n^2+1)^{\frac{2}{3}}}}{\dfrac{1}{n^{\frac{4}{3}}}}$$

$$n^{\frac{4}{3}}\left(1+\frac{5}{n^2}\right)^{\frac{2}{3}} + n^{\frac{2}{3}}\left(1+\frac{5}{n^2}\right)^{\frac{1}{3}} n^{\frac{2}{3}}\left(1+\frac{1}{n^2}\right)^{\frac{1}{3}} + n^{\frac{4}{3}}\left(1+\frac{1}{n^2}\right)^{\frac{2}{3}}$$

$$= \lim_{n\to\infty} \frac{4n^{\frac{4}{3}}}{n^{\frac{4}{3}}\left[\left(1+\frac{5}{n^2}\right)^{\frac{2}{3}} + \left(1+\frac{5}{n^2}\right)^{\frac{1}{3}}\left(1+\frac{1}{n^2}\right)^{\frac{1}{3}} + \left(1+\frac{1}{n^2}\right)^{\frac{2}{3}}\right]} = \frac{4}{3}$$

$$\sum_{n=1}^{\infty} b_n \text{ K} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ K}$$

$$\sum_{n=1}^{\infty} \frac{e^n\sqrt{n}}{3^n} \qquad a_n = \frac{e^n\sqrt{n}}{3^n}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{e^n\sqrt{n}}{3^n}} = \lim_{n\to\infty} \frac{e}{3} \sqrt[n]{\sqrt{n}} = \frac{e}{3} \lim_{n\to\infty} \sqrt{\sqrt[n]{n}} = \frac{e}{3} \sqrt{\lim_{n\to\infty} \sqrt[n]{n}}$$

$$= \frac{e}{3} < 1$$

$$\sum_{n=1}^{\infty} \frac{e^n\sqrt{n}}{3^n} \quad \textbf{konverguje}$$

$$\sum_{n=1}^{\infty} \frac{e^n\sqrt{n}}{3^n} \qquad a_n = \frac{e^n\sqrt{n}}{3^n}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{e^n\sqrt{n}}{3^n}} = \lim_{n\to\infty} \frac{e}{3} \sqrt[n]{\sqrt{n}} = \frac{e}{3} \lim_{n\to\infty} \sqrt[n]{\sqrt{n}} = \frac{e}{3} < 1$$

$$\sum_{n=1}^{\infty} \frac{e^n\sqrt{n}}{3^n} \text{ konverguje}$$

$$1 \le \underset{\to 1}{\sqrt[n]{\sqrt{n}}} \le \underset{\to 1}{\sqrt[n]{n}}$$

$$\sum_{n=1}^{\infty} \frac{1}{1+4^n} \qquad a_n = \frac{1}{1+4^n} \quad a_{n+1} = \frac{1}{1+4^{n+1}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{1+4^{n+1}}}{\frac{1}{1+4^n}} = \lim_{n\to\infty} \frac{1+4^n}{1+4^{n+1}} = \lim_{n\to\infty} \frac{4^n(\frac{1}{4^n}+1)}{4^{n+1}(\frac{1}{4^{n+1}}+1)}$$

$$= \frac{1}{4} \quad < 1$$

LPK

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{1+4^n} \quad \textbf{konverguje}$$

$$\sum_{n=1}^{\infty} \frac{1}{1+4^n} \qquad a_n = \frac{1}{1+4^n}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{1+4^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{1}}{\sqrt[n]{1+4^n}} = \frac{\lim_{n\to\infty} \sqrt[n]{1}}{\lim_{n\to\infty} \sqrt[n]{1+4^n}} = \frac{1}{4} < 1$$

LOK

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{1+4^n} \text{ konverguje}$$

$$4 = \sqrt[n]{4^n} < \sqrt[n]{1+4^n} < \sqrt[n]{4^n+4^n} = \sqrt[n]{2.4^n} = 4\sqrt[n]{2}$$

$$\to 4$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{1+4^n} = 4$$