1. $\lim\limits_{n\to\infty} \sqrt[n]{3^n+n!+n^n+n^3} \cdot \arcsin\frac{1}{n^2} \cdot \arcsin\left(\frac{n^2}{n^2+1}\right) \overset{\text{H.V.}}{=}$

$\overset{\text{VOAL}}{=} \lim\limits_{x\to\infty} \arcsin\left(\frac{x^2}{x^2+1}\right) \cdot \lim\limits_{n\to\infty} \underbrace{\sqrt[n]{n^n+n!+3^n+n^3} \cdot \arcsin\frac{1}{n^2}}_{a_n}$

$= \arcsin 1 \cdot \lim a_n = \frac{\pi}{2}\cdot 0 = \underline{\underline{0}}$

$\underbrace{\arcsin\frac{1}{n^2} \sqrt[n]{n^n}}_{b_n} \leq a_n \leq \underbrace{\sqrt[n]{4\cdot n^n} \cdot \arcsin\frac{1}{n^2}}_{c_n}$

$\lim\limits_{n\to\infty} c_n = \lim \overbrace{\sqrt[n]{4}}^{\to 1} \cdot n \cdot \arcsin\frac{1}{n^2} =$

$= 1\cdot \lim \frac{n}{n^2} \cdot \underbrace{\frac{\arcsin\frac{1}{n^2}}{\frac{1}{n^2}}}_{\longrightarrow 1 \quad (*)} = 1\cdot 1\cdot \lim\frac{1}{n} = 0$

(*) $\lim\limits_{n\to\infty} \frac{\arcsin\frac{1}{n^2}}{\frac{1}{n^2}} \overset{\text{H.V.}}{=} \lim\limits_{x\to\infty} \frac{\arcsin\frac{1}{x^2}}{\frac{1}{x^2}} \overset{\text{VOLSF}}{=} 1$

<u>VOLSF:</u> vnější $f(y) = \frac{\arcsin y}{y}$ $\quad \lim\limits_{y\to \underline{0}} f(y) = 1$

vnitřní $g(x) = \frac{1}{x^2}$ $\quad \lim\limits_{x\to\infty} g(x) = \underline{\underline{0}}$

<u>(P)</u> triv. – dokonce $\forall x\in(0,\infty)$ $g(x) \neq \underline{\underline{0}}$.

<u>Celkem:</u> $0 \leq a_n \leq c_n \to 0 \overset{\text{POLICAJTI}}{\Longrightarrow} \lim a_n = 0.$

(2.) $$\lim_{x\to\frac{\pi}{2}+}(\sin x)^{\frac{1}{\cos^3 x}} = \lim_{x\to\frac{\pi}{2}+} e^{\frac{1}{\cos^3 x}\cdot\ln(\sin x)} \overset{(**)}{=}$$

$$\lim_{x\to\frac{\pi}{2}+}\frac{1}{\cos^3 x}\cdot\ln(\sin x) = \left| \begin{array}{l} y = x-\frac{\pi}{2}\to 0_+ \\ x = y+\frac{\pi}{2} \\ \text{„SUBSTITUCE"} \end{array}\right| =$$

$$= \lim_{y\to 0_+}\frac{\ln\left(\sin\left(y+\frac{\pi}{2}\right)\right)}{\cos^3\left(y+\frac{\pi}{2}\right)} = \lim_{y\to 0_+}\frac{\ln(\cos y)}{(-\sin y)^3} =$$

$$= \lim_{y\to 0_+}\underbrace{\frac{\ln(\cos y)}{\cos y-1}}_{(*)\ \to 1}\cdot\underbrace{\frac{\cos y-1}{-y^2}}_{\xrightarrow{\text{známá}}\ \frac{1}{2}}\cdot\frac{-y^2}{-y^3}\cdot\underbrace{\frac{-y^3}{-\sin^3 y}}_{\xrightarrow{\text{známá + VOAL}}\ 1^3=1}$$

$$\overset{\text{VOAL}}{=} 1\cdot\frac{1}{2}\cdot 1\cdot\lim_{y\to 0_+}\frac{1}{y} = \frac{1}{2}\cdot\infty = \underline{\underline{\infty}}$$

(*) VOLSF: vnější $f(z) = \dfrac{\ln(z)}{z-1}$, $\lim_{z\to 1} f(z) \overset{\text{známá}}{=} 1$

vnitřní $g(y) = \cos y$, $\lim_{y\to 0_+}\cos y = \underline{1}$

(P) víme, že platí např. pro $\delta = \frac{\pi}{2}$, tj.

$$\forall y\in P_+\left(0,\tfrac{\pi}{2}\right):\ \cos y \neq \underline{\underline{1}}.$$

$$\overset{(**)}{=} \lim_{y\to\infty} e^y \text{ „}= e^{\infty}\text{"} = \infty$$

(3) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{\sqrt{n}}\right) =: \sum_{n=1}^{\infty} a_n$

<u>AK:</u> $|a_n| = |(-1)^n| \cdot \left|\sin\left(\frac{1}{\sqrt{n}}\right)\right| = 1 \cdot \left|\sin\left(\frac{1}{\sqrt{n}}\right)\right| =$

$= \sin \frac{1}{\sqrt{n}}$, protože $\forall n \in \mathbb{N}: \frac{1}{\sqrt{n}} \in [0, \pi]$

a $\sin \geq 0$ na $[0, \pi]$.

$\displaystyle\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \sin \frac{1}{\sqrt{n}}$ $\nearrow 0$ srovnáme s řadou $\displaystyle\sum_{n=1}^{\infty} b_n$,

kde $b_n := \frac{1}{\sqrt{n}}$; ta D.

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \overset{(*)}{=} 1 \in (0, \infty).$$

Podle LSK tedy $\left(\sum |a_n|\ K. \Leftrightarrow \sum b_n\ K.\right)$, ovšem

$\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ D., a tedy $\sum |a_n|$ D.

<u>K:</u> Protože $\sqrt{n}$ je <u>rostoucí</u>, je $\frac{1}{\sqrt{n}}$ <u>klesající</u>.

Navíc $\forall n \in \mathbb{N}: \frac{1}{\sqrt{n}} \in (0, \frac{\pi}{2})$ a sin je na $(0, \frac{\pi}{2})$ <u>rostoucí</u>. <u>Celkem:</u> $\left\{\sin\left(\frac{1}{\sqrt{n}}\right)\right\}$ je <u>klesající</u>.

Přitom $\displaystyle\lim_{n\to\infty} \sin\frac{1}{\sqrt{n}} = \sin 0 = 0$.

Podle Leibnizova kritéria tedy $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{\sqrt{n}}\right)$ KONVERGUJE. <u>Závěr:</u> $\displaystyle\sum_{n=1}^{\infty} a_n$ RK.

4. $f(x) = \sqrt[3]{x^3 - x^2 - 2x}$ ... 1b • $D_f = \mathbb{R}$, f je spoj. na $\mathbb{R}$

1b • symetrie nejsou • $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty$

3b • asymptoty: $a := \lim\limits_{x\to\infty} \frac{f(x)}{x} = \lim\limits_{x\to\infty} \sqrt[3]{1 - \frac{1}{x} - \frac{2}{x^2}} = 1$

$$b := \lim_{x\to\infty} \Big(f(x) - \overbrace{x}^{a\cdot x = x}\Big) = \lim_{x\to\infty}\left(\sqrt[3]{x^3-x^2-2x} - x\right) =$$

$$= \lim_{x\to\infty} \frac{x^3 - x^2 - 2x - x^3}{(x^3-x^2-2x)^{2/3} + (x^3-x^2-2x)^{1/3}\,x + x^2} = \lim_{x\to\infty} \frac{-x^2-2x}{x^2\left((1-\dots)^{2/3} + (1-\dots)^{1/3}\cdot 1 + 1\right)}$$

$$= \lim \frac{1}{\underbrace{(\;)}_{\to 1} + \underbrace{(\;)}_{\to 1} + 1} \cdot \lim_{x\to\infty} \frac{-x^2-2x}{x^2} = \frac{1}{3}\cdot \lim \frac{-1-\frac{2}{x}}{1} = -\frac{1}{3}$$

Výpočet projde zcela stejně i pro $x \to -\infty$. Tedy

$A(x) = ax + b = x - \frac{1}{3}$ je asymptota v $+\infty$ i $-\infty$.

2b $f'(x) = \left((x^3 - x^2 - 2x)^{\frac{1}{3}}\right)' = \frac{1}{3}(x^3-x^2-2x)^{-\frac{2}{3}} \cdot (3x^2 - 2x - 2)$

kořeny polynomů: • $x^3 - x^2 - 2x = x\cdot(x^2 - x - 2) = x\cdot(x-2)(x+1)$

• $3x^2 - 2x - 2 = 0$ ... $x_{1,2} = \frac{2 \pm \sqrt{4+24}}{6} = \frac{2 \pm 2\sqrt{7}}{6} = \left\langle \begin{array}{l} \frac{1+\sqrt{7}}{3} \\ \frac{1-\sqrt{7}}{3} \end{array}\right.$

Tedy: 1b • vzorec pro $f'(x)$ platí pro $x \in \mathbb{R} \setminus \{-1, 0, 2\}$.

• znaménko $f'(x)$ nezávisí na $\frac{1}{3}(x^3 - x^2 - 2x)^{-\frac{2}{3}}$ (vždy kladné)

tj. $f'(x) > 0 \iff (3x^2 - 2x - 2) > 0 \iff x \in \left(-\infty, \frac{1-\sqrt{7}}{3}\right) \vee x \in \left(\frac{1+\sqrt{7}}{3}, \infty\right)$.

2b

Celkem:

	$\left(-\infty, \frac{1-\sqrt{7}}{3}\right)$	$\left(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right)$	$\left(\frac{1+\sqrt{7}}{3}, \infty\right)$
f'	$\oplus$	$\ominus$	$\oplus$
f	$\nearrow$	$\searrow$	$\nearrow$

$-1 < \frac{1-\sqrt{7}}{3} < 0 < \frac{1+\sqrt{7}}{3} < 2$

1b

V bodě $\frac{1-\sqrt{7}}{3}$ je tedy lokální max. f,

$\frac{1+\sqrt{7}}{3}$ min f.

$$f\left(\frac{1-\sqrt{7}}{3}\right) = \sqrt[3]{\left(\frac{1-\sqrt{7}}{3}\right)^3 - \left(\frac{1-\sqrt{7}}{3}\right)^2 - 2\left(\frac{1-\sqrt{7}}{3}\right)} = \ldots$$

2b

Derivace v bodech $-1, 0, 2$: f je v těchto b. spojitá (dokonce na $\mathbb{R}$), takže užijeme větu o lim. derivace:

- $$\lim_{x\to -1} f'(x) = \lim_{x\to -1} \frac{1}{3}\left(x^3 - x^2 - 2x\right)^{-\frac{2}{3}}\left(3x^2 - 2x - 2\right) =$$
$$= \frac{1}{3}\left(3(-1)^2 - 2(-1) - 2\right) \cdot \lim_{x\to -1} \frac{1}{\left(\sqrt[3]{x^3 - x^2 - 2x}\right)^2} = \text{"}\frac{3}{3} \cdot \frac{1}{0_+}\text{"} = \infty$$

- $\lim_{x\to 0} f'(x) = \text{"podobně"} = -\infty$
- $f'(2) = \lim_{x\to 2} f'(x) = \ldots = \infty$.

Celkem: $f'(-1) = \infty$, $f'(0) = -\infty$, $f'(2) = \infty$.

2. DERIVACE „NEBYLA POTŘEBA" – pouze 2b

2b

$$f''(x) = \left(\frac{1}{3}\left(x^3 - x^2 - 2x\right)^{-\frac{2}{3}}\left(3x^2 - 2x - 2\right)\right)' =$$
$$= \frac{1}{3} \cdot \frac{-2}{3} \cdot \left(x^3 - x^2 - 2x\right)^{-\frac{5}{3}}\left(3x^2 - 2x - 2\right)^2 +$$
$$+ \frac{1}{3} \cdot \left(x^3 - x^2 - 2x\right)^{-\frac{2}{3}}\left(6x - 2\right) =$$
$$= \frac{1}{3} \cdot \left(x^3 - x^2 - 2x\right)^{-2/3} \cdot \left(-\frac{2}{3} \cdot \frac{\left(3x^2 - 2x - 2\right)^2}{x^3 - x^2 - 2x} + (6x - 2)\right) =$$
$$= \frac{1}{3}(\ldots)^{-2/3} \cdot \frac{1}{\ldots}\left(-\frac{2}{3}\left(9x^4 + 4x^2 + 4 - 12x^3 - 12x^2 + 8x\right) + \cancel{6x^4} - 6x^3 - \cancel{12x^2}\right.$$
$$\left. - 2x^3 + 2x^2 + 4x\right) = \frac{1}{3}(\ldots)^{-2/3} \cdot \frac{1}{\ldots}\left(\cancel{2x^3} + \left(\frac{2}{3}\cdot 8 - 10\right)x^2 + \left(-\frac{2}{3}\cdot 8 + 4\right)x - \cancel{2x^3}\right.$$
$$\left. - \frac{8}{3}\right) = \frac{1}{3} \cdot \frac{(\ldots)^{-2/3}}{(\ldots)} \cdot \left(\frac{-14}{3}x^2 - \frac{4}{3}x - \frac{8}{3}\right) = \frac{-2}{9} \cdot (\ldots)^{-5/3}\left(+7x^2 + 2x + 4\right)$$

$$f''(x) = -\frac{2(7x^2+2x+4)}{9\cdot(x^3-x^2-2x)^{5/3}}.$$

Polynom v čitateli je kladný pro každé $x \in \mathbb{R}$ (neboť $D = 2^2 - 4\cdot 7\cdot 4 < 0$), takže znaménko f'' ovlivňuje pouze $(x^3 - x^2 - 2x) = x\cdot(x-2)(x+1)$. Celkem:

	$(-\infty,-1)$	$(-1,0)$	$(0,2)$	$(2,\infty)$
f''	$+$	$-$	$+$	$-$
f	$\cup$	$\cap$	$\cup$	$\cap$

$-1, 0, 2$ jsou <u>inflexní body</u> f.