① $\lim\limits_{n\to\infty} \operatorname{arctg}(-n) \cdot$

$$\cdot\left(\sqrt{n^4 + n^2\ln(\ln n)} - \sqrt{n^4 - n\sin n}\right)$$

$$\overset{\text{VOAL}}{\underset{\text{(H.V.)}}{=}} -\frac{\pi}{2} \cdot \lim_{n\to\infty} \frac{n^4 + n^2\ln(\ln n) - (n^4 - n\cdot\sin n)}{\sqrt{n^4\left(1+\frac{\ln(\ln n)}{n^2}\right)} + \sqrt{n^4\cdot\left(1-\frac{\sin n}{n^3}\right)}}$$

$$= -\frac{\pi}{2} \cdot \lim_{n\to\infty} \frac{\cancel{n^2}\left(\ln(\ln n) + \frac{\sin n}{n}\right)}{\cancel{n^2}\left(\sqrt{1+\frac{\ln(\ln n)}{n^2}} + \sqrt{1-\frac{\sin n}{n^3}}\right)}$$

$$\overset{\text{VOAL}}{=} -\frac{\pi}{2} \cdot \frac{\lim \ln(\ln n) + \lim \frac{\sin n}{n}}{\lim\sqrt{\cdots} + \lim\sqrt{\cdots}} \overset{(*)}{=}$$

$$\overset{(*)}{=} -\frac{\pi}{2} \cdot \frac{\infty + 0}{\sqrt{1+0} + \sqrt{1-0}} = -\frac{\pi}{2} \cdot \frac{\infty}{2} = \underline{\underline{-\infty}}$$

<u>H.V.</u>: $f(y) = \operatorname{arctg} y$, $\lim\limits_{y\to-\infty} f(y) = -\frac{\pi}{2}$

$x_n := -n$, $\lim\limits_{n\to\infty} x_n = -\infty$.

<u>(*)</u>: Používáme • „$\lim\sqrt{\ } = \sqrt{\lim}$",

• „nulová · omezená = nulová":

$\{\sin n\}$ je omezená posl., $\frac{1}{n^3} \to 0$.

• „srovnávací škála" $\rightsquigarrow$ $\lim\limits_{n\to\infty} \frac{\ln(\ln n)}{n^2} = 0$.

② $$\lim_{x\to\infty}\left(\sqrt[3]{\frac{5x+4}{5x-3}}\right)^{\frac{x^2+1}{x+1}} = \lim_{x\to\infty}\exp\left[\frac{x^2+1}{x+1}\cdot\ln\left(\frac{5x+4}{5x-3}\right)^{\frac{1}{3}}\right] \underset{\text{VOLSF (S)}}{=} e^{\frac{7}{15}}$$

$$\lim_{x\to\infty}\frac{x^2+1}{x+1}\cdot\frac{1}{3}\ln\frac{5x-3+7}{5x-3} = \frac{1}{3}\lim_{x\to\infty}\frac{x^2+1}{x+1}\cdot\frac{\ln\left(1+\frac{7}{5x-3}\right)}{\frac{7}{5x-3}}\cdot\frac{7}{5x-3}$$

$$\overset{\text{VOLSF}}{\underset{\text{VOAL}}{=}} \frac{1}{3}\cdot 1\cdot 7\cdot\lim_{x\to\infty}\frac{\cancel{x^2}\left(1+\frac{1}{x^2}\right)}{\cancel{x}\left(1+\frac{1}{x}\right)\cdot\cancel{x}\left(5-\frac{3}{x}\right)} = \frac{7}{3}\cdot\frac{1+0}{(1+0)\cdot(5-0)} = \frac{7}{3\cdot 5} = \frac{7}{15}$$

VOLSF (S): $f(y) = \exp y = e^y$ je spojitá na $\mathbb{R}$, a tedy i v bodě $y = \frac{7}{15}$.

VOLSF (P): $f(y) = \frac{\ln(1+y)}{y}$, $\lim_{y\to 0} f(y) \overset{\text{„známá“}}{=} 1$

vnitřní fce $g(x) = \frac{7}{5x-3}$, $\lim_{x\to\infty} g(x) = 0$

(P): $\delta := 1$; pak $\forall x\in P(\infty,\delta): g(x)\neq 0$.

Tedy $\lim f\circ g(x) = 1$.

③ $\sum_{n=1}^{\infty} n^n \cdot \sin\left(\frac{4^n}{3^{2n} \cdot n!}\right) =: \sum_{n=1}^{\infty} a_n$

$\lim_{n\to\infty} \frac{4^n}{3^{2n} \cdot n!} = \lim \frac{4^n}{9^n \cdot n!} = 0,$

tj. argument sinu jde do 0.

Srovnáme tedy s řadou $\sum_{n=1}^{\infty} b_n$, kde

$b_n := n^n \cdot \frac{4^n}{3^{2n} \cdot n!}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{n^n} \cdot \sin\left(\frac{4^n}{3^{2n} \cdot n!}\right)}{\cancel{n^n} \cdot \frac{4^n}{3^{2n} \cdot n!}} \overset{H.V.}{=} 1$$

H.V.: $f(y) = \frac{\sin y}{y}$, $\lim_{y\to 0} f(y) \overset{\text{"známá"}}{=} 1$

$x_n := \frac{4^n}{3^{2n} \cdot n!}$ ... (H1) $\lim x_n = 0$ ✓

(H2) $\forall n \in \mathbb{N}: x_n \neq 0$ ✓.

LSK $\Rightarrow$ [$\sum a_n$ K. $\Leftrightarrow$ $\sum b_n$ K.]

$\sum b_n$ K. ? Použijeme lim. podílové kr.:

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}\, 4^{n+1}}{9^{n+1} \cdot (n+1)!} \cdot \frac{9^n \cdot n!}{n^n \cdot 4^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n \cdot (n+1) \cdot 4}{n^n \cdot (n+1) \cdot 9} = \frac{4}{9} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

$$= \frac{4e}{9} > \frac{4 \cdot \frac{5}{2}}{9} = \frac{10}{9} > 1.$$

Tedy $\sum b_n$ D. [Tedy $\sum a_n$ D.]

4. $f(x) = x \cdot e^{-x^2}$ ... • $D_f = \mathbb{R}$, spojitá.

• $f(-x) = -x \cdot e^{-(-x)^2} = -x e^{-x^2} = -f(x)$,

funkce je tedy <u>lichá</u>.

• $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x}{e^{x^2}} \overset{L'H}{=} \lim \dfrac{1}{2xe^{x^2}} = 0$

$e^{x^2} \to \infty$

lichost $\rightsquigarrow \lim\limits_{x\to-\infty} f(x) = 0$.

• $f'(x) = e^{-x^2} + x e^{-x^2} \cdot (-2x) =$

$= e^{-x^2}(1-2x^2) = 0 \iff$

$1-2x^2 = 0 \iff x^2 = \frac{1}{2} \iff x \in \left\{\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right\}$

$x \in$	$(-\infty, -\sqrt{\frac{1}{2}})$	$(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$	$(\sqrt{\frac{1}{2}}, \infty)$
f'	$\ominus$	$\oplus$	$\ominus$
f	$\searrow$	$\nearrow$	$\searrow$

• Lokální & globální minimum v bodě $-\sqrt{\frac{1}{2}}$:

$f(-\sqrt{\frac{1}{2}}) = -\sqrt{\frac{1}{2}} \cdot e^{-\frac{1}{2}}$. Symetricky:

lok. & glob max. v bodě $\sqrt{\frac{1}{2}}$

$f(\sqrt{\frac{1}{2}}) = \sqrt{\frac{1}{2}} e^{-\frac{1}{2}}$.

• $H_f = \left[-\sqrt{\frac{1}{2}} e^{-\frac{1}{2}}, \sqrt{\frac{1}{2}} e^{-\frac{1}{2}}\right] = \left[-\sqrt{\frac{1}{2e}}, \sqrt{\frac{1}{2e}}\right]$

$$f'(x) = e^{-x^2}(1-2x^2)$$

- $$f''(x) = e^{-x^2}\cdot(-2x)(1-2x^2) + e^{-x^2}(-4x) =$$

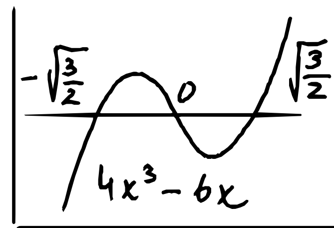
$$= e^{-x^2}(+4x^3 - 2x - 4x) =$$

$$= e^{-x^2}(4x^3 - 6x) = 0 \iff$$

$$4x^3 - 6x = 0 \iff x(2x^2-3) = 0 \iff$$

$$x = 0 \ \vee \ x^2 = \frac{3}{2} \iff x \in \left\{-\sqrt{\frac{3}{2}}, 0, \sqrt{\frac{3}{2}}\right\}$$

Inflexní body jsou tedy 3:



Tabulka:

$x\in$	$(-\infty, -\sqrt{\frac{3}{2}})$	$(-\sqrt{\frac{3}{2}}, 0)$	$(0, \sqrt{\frac{3}{2}})$	$(\sqrt{\frac{3}{2}}, \infty)$
f''	$-$	$+$	$-$	$+$
f	⌢	⌣	⌢	⌣