1. $$\lim_{n\to\infty} \frac{(-1)^n n^8 + 8^n + n! + \ln(8n)}{\ln(8^n) + (n+1)! + 9n^9 + \sin(n^n)} \cdot \operatorname{arctg} n \cdot \ln(e^n+1) =$$

$$= \lim_{n\to\infty} \operatorname{arctg} n \cdot \lim_{n\to\infty} \frac{n! \cdot \left(\frac{(-1)^n n^8}{n!} + \frac{8^n}{n!} + 1 + \frac{\ln(8n)}{n!}\right)}{(n+1)! \cdot \left(\frac{n \cdot \ln 8}{(n+1)!} + 1 + \frac{9n^9}{(n+1)!} + \frac{\sin(n^n)}{(n+1)!}\right)} \cdot \ln\left(e^n\left(1+\frac{1}{e^n}\right)\right)$$

$$= \frac{\pi}{2} \cdot \lim \frac{\dots + \dots + 1 + \dots}{\dots + 1 + \dots + \dots} \cdot \lim \frac{\ln e^n + \ln\left(1+\frac{1}{e^n}\right)}{n+1} =$$

$$= \frac{\pi}{2} \cdot \frac{0+0+1+0}{0+1+0+0} \cdot \lim \left(\frac{n}{n+1} + \frac{\ln\left(1+\frac{1}{e^n}\right)}{n+1}\right) =$$

$$= \frac{\pi}{2} \cdot (1+0) = \frac{\pi}{2}$$

Použili jsme: srovnávací škálu: $\frac{n^8}{n!}$, $\frac{8^n}{n!}$, $\frac{\ln 8n}{n!}$, $\frac{n \cdot \ln 8}{n!}$, $\frac{n^9}{(n+1)!}$ mají limitu 0. Podle „nulová · omezená" tedy i

$$\lim_{n\to\infty} \frac{(-1)^n \cdot n^8}{n!} = 0 \quad \text{a} \quad \lim \frac{\sin(n^n)}{(n+1)!} = 0.$$

2. $$\lim_{x\to 4} \frac{\sin x \cdot (e^{x-4}-1)}{\sqrt{x^2+20} - \sqrt{4x+20}} = \sin 4 \cdot \lim_{y\to 0} \frac{e^y - 1}{\sqrt{(y+4)^2+20} - \sqrt{4y+36}} =$$

$$\left[\begin{array}{l} y = x-4 \to 0 \\ x = y+4 \end{array}\right] = \sin 4 \cdot \lim_{y\to 0} \frac{e^y-1}{y} \cdot \lim_{y\to 0} \frac{y \cdot \left(\sqrt{(y+4)^2+20} + \sqrt{4y+36}\right)}{y^2+8y+36-(4y+36)}$$

$$= \sin 4 \cdot 1 \cdot \lim \left(\sqrt{\ } + \sqrt{\ }\right) \cdot \lim_{y\to 0} \frac{y}{y^2+4y} =$$

$$= \sin 4 \cdot \left(\sqrt{36} + \sqrt{36}\right) \cdot \lim_{y\to 0} \frac{1}{y+4} = \sin 4 \cdot 12 \cdot \frac{1}{4} = 3 \cdot \sin 4$$

(3.) $$\sum_{n=1}^{\infty} \frac{\ln(n^3+1+\sqrt{n}) - \ln(n^3+1)}{\sin\frac{1}{n^2}} =$$

$$= \sum_{n=1}^{\infty} \frac{\ln\left(\frac{n^3+1+\sqrt{n}}{n^3+1}\right)}{\sin\frac{1}{n^2}} = \sum_{n=1}^{\infty} \frac{\ln\left(1+\frac{\sqrt{n}}{n^3+1}\right)}{\sin\frac{1}{n^2}}$$

Srovnáme s řadou $\sum_{n=1}^{\infty} b_n$, kde $b_n := \frac{\frac{\sqrt{n}}{n^3}}{\frac{1}{n^2}} = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \overset{(+)}{\frac{\ln\left(1+\frac{\sqrt{n}}{n^3+1}\right)}{\frac{\sqrt{n}}{n^3+1}}} \cdot \frac{\frac{\sqrt{n}}{n^3+1}}{\frac{\sqrt{n}}{n^3}} \cdot \frac{\frac{1}{n^2}}{\sin\frac{1}{n^2}} =$$

Heineho věta: • $f(y) = \frac{\ln(y+1)}{y}$, $\lim_{y\to 0} f(y) \overset{\text{známá}}{=} 1$

(H1) $a_n = \frac{\sqrt{n}}{n^3+1} \xrightarrow{n\to\infty} 0$

(H2) zřejmě $a_n \neq 0$ pro všechna $n$

$\Longrightarrow \lim_{n\to\infty} (+) = 1$

Podobně H.V. pro: $\lim_{n\to\infty} \frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}} = 1.$

$$= 1 \cdot \frac{1}{1} \cdot \lim \frac{n^3}{n^3+1} = 1 \in (0,\infty).$$

Tedy podle lim. srov. kritéria: ($\sum a_n$ K $\Leftrightarrow$ $\sum b_n$ K)

Tedy (protože $\sum b_n$ D) $\sum a_n$ D

(4.) $f(x) = |x-2| - 2\,\mathrm{arctg}\,x$ $\quad \ldots 1 \quad D_f = \mathbb{R}$, spojitá.

1 • symetrie (sudost / lichost / periodicita) nejsou.

• $\lim\limits_{x\to\infty} f(x) = \lim ((x-2) - 2\,\mathrm{arctg}\,x) = \infty - 2 - \pi = \infty$

1 $\lim\limits_{x\to-\infty} f(x) = \ldots = -\infty$

• asymptoty: $\underline{v + \infty:}$ $\quad a := \lim\limits_{x\to\infty} \frac{f(x)}{x} = $ zřejmě $= 1$

3 $\quad b := \lim\limits_{x\to\infty} (f(x) - ax) = \lim\limits_{x\to\infty} \left((x-2) - 2\,\mathrm{arctg}\,x - 1\cdot x\right) =$

$= \lim (-2 - 2\,\mathrm{arctg}\,x) = -2 - \pi$

$\Rightarrow$ asymptota v $+\infty$ je $y = ax + b = x - 2 - \pi$.

$\underline{v - \infty:}$ $\quad a := \lim\limits_{x\to-\infty} \frac{f(x)}{x} = \lim \left((2-x) - 2\,\mathrm{arctg}\,x\right)\cdot\frac{1}{x} = \ldots = -1$

$b := \lim\limits_{x\to-\infty} (f(x) - ax) = \lim\limits_{x\to-\infty} \left((2-x) - 2\,\mathrm{arctg}\,x + x\right) = 2 + \pi$

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• $\underline{\text{derivace}}$ $\quad f'(x) = \mathrm{sgn}(x-2) - \frac{2}{1+x^2} = $ pro $x \neq 2$

2 $= \begin{cases} x>2: & \frac{x^2-1}{x^2+1} > 0 \iff x^2 - 1 > 0 \iff x \in (-\infty,-1) \cup (1,\infty) \\ x<2: & \frac{-x^2-3}{x^2+1} > 0 \iff -x^2 - 3 > 0 \iff +x^2 + 3 < 0 \iff x \in \emptyset. \end{cases}$

2 Tabulka:

| | $(-\infty,2)$ | $(2,\infty)$ |
|---|---|---|
| $f$ | $\searrow$ ~~$(-\infty,2)$~~ | $\nearrow$ |
| $f'$ | $\ominus$ | $\oplus$ |

$\Rightarrow$ globální (i lokální) $\underline{\text{minimum}}$

$f(2) = -2\,\mathrm{arctg}\,2$

3 • $\underline{\text{2. der}}$: $\quad f''(x) = \left(\mathrm{sgn}(x-2) - \frac{2}{1+x^2}\right)' =$

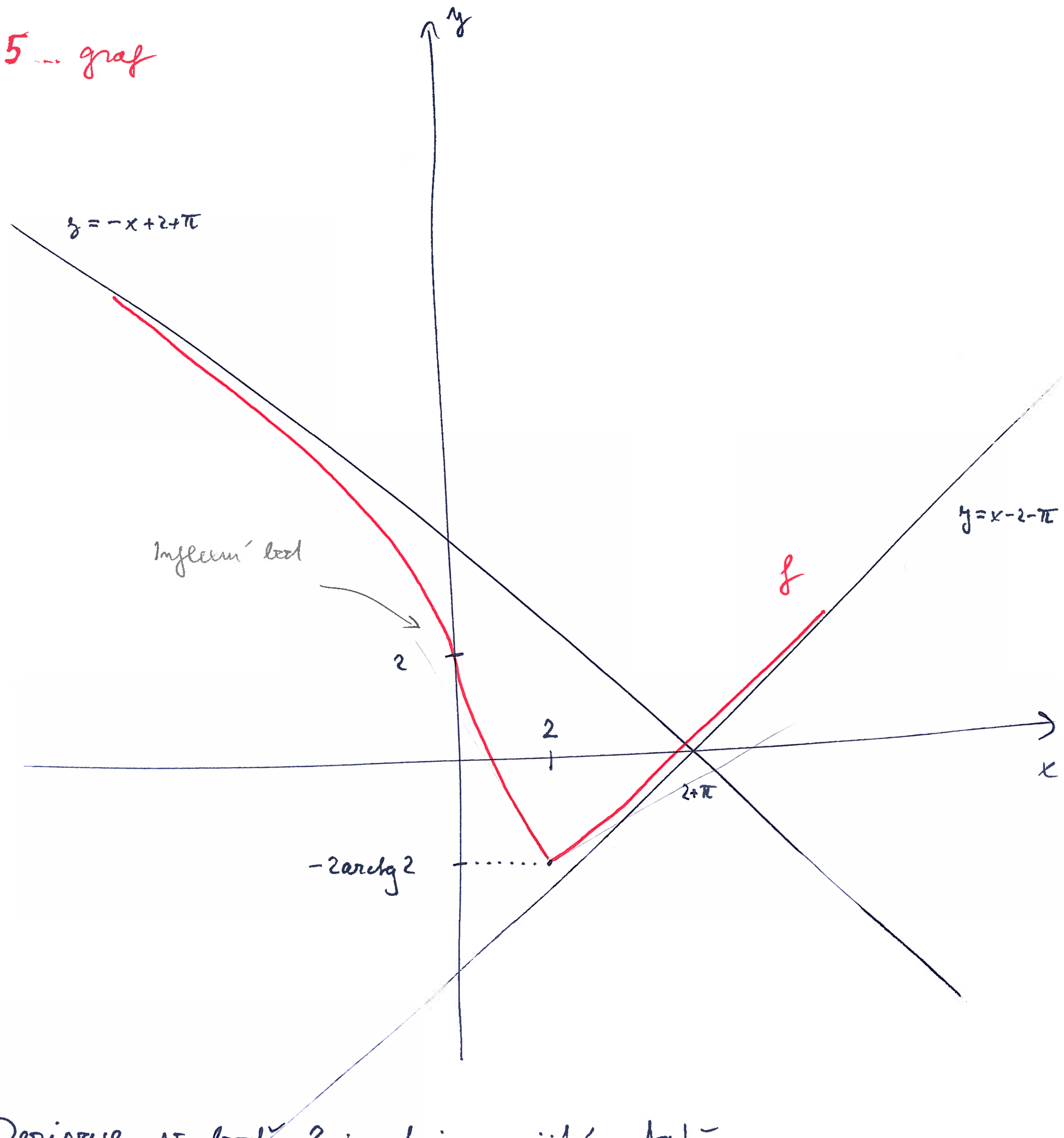
2

$= 0 - \left(\frac{-2\cdot 2x}{(1+x^2)^2}\right) = \frac{4x}{(1+x^2)^2} \Rightarrow$

1

| | $(-\infty,0)$ | $(0,2)$ | $(2,\infty)$ |
|---|---|---|---|
| $f''$ | $\ominus$ | $\oplus$ | $\oplus$ |
| $f$ | $\cap$ | $\cup$ | $\cup$ |

5 ... graf




• Derivace v bodě 2: f je spojitá, takže

2 $f'_+(2) = \lim_{x\to 2+} f'(x)$, má-li P.S.S. a podobně zleva.

Tedy: $f'_+(2) = \lim_{x\to 2+} \frac{x^2-1}{x^2+1} = \frac{4-1}{4+1} = \frac{3}{5}$

$f'_-(2) = \lim_{x\to 2-} \frac{-x^2-3}{x^2+1} = \frac{-4-3}{4+1} = \frac{-7}{5}$