1. $$\lim_{n\to\infty}\left(\sqrt[3]{n^3-6n}-\sqrt[3]{n^3-6}\right)\cdot\ln\left(n^4+4^n\right)=$$

3b

$$=\lim_{n\to\infty}\frac{n^3-6n-(n^3-6)}{(n^3-6n)^{\frac{2}{3}}+(n^3-6n)^{\frac{1}{3}}(n^3-6)^{\frac{1}{3}}+(n^3-6)^{\frac{2}{3}}}\cdot\ln\left(4^n\cdot\left(1+\frac{n^4}{4^n}\right)\right)=$$

2b

$$=\lim_{n\to\infty}\frac{6-6n}{(\,)+(\,)(\,)+(\,)}\cdot\left(n\cdot\ln 4+\ln\left(1+\frac{n^4}{4^n}\right)\right)=$$

$$=6\cdot\lim_{n\to\infty}\frac{(1-n)\cdot\left(n\cdot\ln 4+\ln\left(1+\frac{n^4}{4^n}\right)\right)}{\left(n^3\left(1-\frac{6}{n^2}\right)\right)^{\frac{2}{3}}+\left(n^3\left(1-\frac{6}{n^2}\right)\right)^{\frac{1}{3}}\left(n^3\left(1-\frac{6}{n^3}\right)\right)^{\frac{1}{3}}+\left(n^3\left(1-\frac{6}{n^3}\right)\right)^{\frac{2}{3}}}=$$

1b, 2b

$$=6\cdot\lim\frac{\cancel{n^2}\cdot\left(\frac{1}{n}-1\right)\cdot\left(\ln 4+\frac{1}{n}\ln\left(1+\frac{n^4}{4^n}\right)\right)}{\cancel{n^2}\cdot\left(\left(1-\frac{6}{n^2}\right)^{\frac{2}{3}}+\left(1-\frac{6}{n^2}\right)^{\frac{1}{3}}\left(1-\frac{6}{n^3}\right)^{\frac{1}{3}}+\left(1-\frac{6}{n^3}\right)^{\frac{2}{3}}\right)}\overset{\text{VOAL}}{=}$$

2b

$$=6\cdot\frac{(0-1)(\ln 4+0\cdot 0)}{(1-0)^{\frac{2}{3}}+(1-0)^{\frac{1}{3}}(1-0)^{\frac{1}{3}}+(1-0)^{\frac{2}{3}}}=\frac{-6\ln 4}{1+1+1}=\underline{\underline{-2\ln 4}}=-4\ln 2$$

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2. $$\lim_{x\to 1}\frac{\sin(x^4-x^3-x+1)\cdot\ln x}{x^3-3x^2+3x+1}=$$

3b, 2b

$$=\lim_{x\to 1}\underbrace{\frac{\sin(x^4-x^3-x+1)}{x^4-x^3-x+1}}_{\overset{(*)}{\to}1}\cdot\frac{x^4-x^3-x+1}{(x-1)^3}\cdot\underbrace{\frac{\ln(x)}{x-1}}_{\to 1\ (\text{známá lim.})}\cdot(x-1)\overset{\text{VOLSF + VOAL}}{=}$$

VOLSF: $f(y)=\frac{\sin y}{y}\xrightarrow{y\to 0}1$, $g(x)=y=x^4-x^3-x+1\xrightarrow{x\to 1}0$

1b

(P): vnitřní fce g je polynom $\Rightarrow$ $g(x)=0$ nastává pro konečně mnoho reálných $x\in\mathbb{R}$ $\Rightarrow$ $\exists\delta>0\ \forall x\in P(1,\delta): g(x)\neq 0$.

$$x^4-x^3-x+1=x^3(x-1)-(x-1)=(x-1)(x^3-1)=(x-1)^2(x^2+x+1)$$

$$=1\cdot 1\cdot\lim_{x\to 1}\frac{(x-1)^2(x^2+x+1)\cdot(x-1)}{(x-1)^3}=\lim_{x\to 1}(x^2+x+1)=\underline{\underline{3}}$$

3b

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}\cdot\left(\frac{2n+1}{n}\right)^{n^2+n}}{(n^2+1)^n\cdot\left(\frac{2n+2}{n}\right)^{n^2}} =: \sum_{n=1}^{\infty} a_n.$

Vzhledem k přítomnosti n-tých mocnin použijeme C. odmocninové kritérium. (Kdybychom měli ještě n! apod., bylo by to spíš na podílové kr.).

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^2\cdot\left(\frac{2n+1}{n}\right)^{n+1}}{(n^2+1)\cdot\left(\frac{2n+2}{n}\right)^{n}} =$$

$$\overset{\text{VOAL}}{=} \lim \frac{n^2}{n^2(1+\frac{1}{n^2})}\cdot \lim\frac{2^{n+1}\cdot\left(\frac{n+\frac{1}{2}}{n}\right)^{n+1}}{2^n\cdot\left(\frac{n+1}{n}\right)^n} =$$

$$\overset{\text{VOAL}}{=} \frac{1}{1+0}\cdot 2\cdot\frac{\lim\left(1+\frac{1/2}{n}\right)^{n+1}}{\lim\left(1+\frac{1}{n}\right)^n} \overset{\text{VOAL}}{=} \frac{2}{e}\cdot\lim\left(1+\frac{1/2}{n}\right)\cdot\lim\left(1+\frac{1/2}{n}\right)^n$$

$$\overset{(*)}{=} \frac{2}{e}\cdot 1\cdot e^{\frac{1}{2}} = \frac{2}{\sqrt{e}} > 1, \text{ a tedy } \sum_{n=1}^{\infty} a_n \text{ D.}$$

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$(*)$: $$\lim_{n\to\infty}\left(1+\frac{1/2}{n}\right)^n \overset{\text{H.V.}}{=} \lim_{x\to\infty} e^{x\cdot\ln\left(1+\frac{1/2}{x}\right)} = \underline{\underline{e^{\frac{1}{2}}}}$$

$$\lim_{x\to\infty} x\cdot\ln\left(1+\frac{1/2}{x}\right) = \lim\frac{\ln\left(1+\frac{1/2}{x}\right)}{\frac{1}{x}\cdot\frac{1}{2}}\cdot\frac{1}{2} =$$

$$= \frac{1}{2}\cdot\lim\frac{\ln\left(1+\frac{1/2}{x}\right)}{\frac{1/2}{x}} \overset{\text{VOLSF}}{=} \frac{1}{2}\cdot 1$$

VOLSF: $f(y)=\frac{\ln(1+y)}{y}$, $g(x)=\frac{1/2}{x}\xrightarrow{x\to\infty}0$ (P) triviálně: $g\neq 0$.

4. $f(x) = \operatorname{arctg}\left(\frac{x+2}{x}\right)$ pro $x \neq 0$, $f(0) := \frac{\pi}{2}$.

Podle def. je $D_f = \mathbb{R}$, spojitá ve všech b. $\mathbb{R} \setminus \{0\}$.

$$\lim_{x \to 0_+} f(x) = \lim_{x \to 0_+} \operatorname{arctg}\Big(\underbrace{\frac{x+2}{x}}_{\text{„}\frac{2}{0_+}\text{“} = +\infty}\Big) = \text{„}\operatorname{arctg} \infty\text{“} = \frac{\pi}{2}.$$

2b

$$\lim_{x \to 0_-} f(x) = \ldots = \text{„}\operatorname{arctg}(-\infty)\text{“} = -\frac{\pi}{2}.$$

Tedy: $\lim\limits_{x \to 0_+} f(x) = f(0) \neq \lim\limits_{x \to 0_-} f(x)$, takže

1b

f je v nule spojitá zprava, ale ne zleva.

1b $\lim\limits_{x \to \pm\infty} \operatorname{arctg}\left(\frac{x+2}{x}\right) = \operatorname{arctg} 1 = \frac{\pi}{4}. \Rightarrow$ Přímka $A(x) = 0 \cdot x + \frac{\pi}{4}$ je asymptota v $\pm\infty$.

Funkce není sudá ani lichá, není periodická.

$$f'(x) = \frac{1}{1 + \left(\frac{x+2}{x}\right)^2} \cdot \frac{1 \cdot x - (x+2) \cdot 1}{x^2} = \frac{1}{\frac{x^2 + x^2 + 4x + 4}{x^2}} \cdot \frac{-2}{x^2} =$$

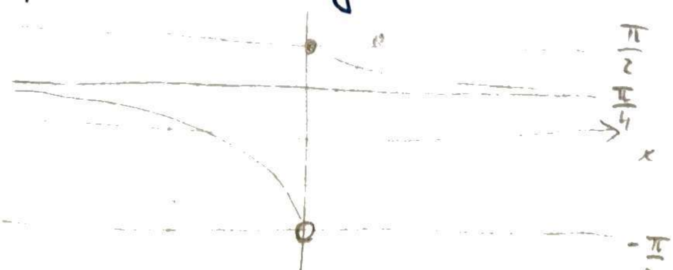
3b

$$= \frac{-2}{2x^2 + 4x + 4} = \frac{-1}{x^2 + 2x + 2} = \frac{-1}{(x+1)^2 + 1}$$ (1b za úpravu), $x \in \mathbb{R} \setminus \{0\}$.

Jmenovatel vždy > 0, takže $f' < 0$ na $(-\infty, 0)$ i $(0, \infty)$:

	$(-\infty, 0)$	$(0, \infty)$
f'	$\ominus$	$\ominus$
f	$\searrow$	$\searrow$

2b



Tedy v bodě 0 je globální (i lokální) ostré maximum, minimum zřejmě není.

Derivace v bodě 1: zřejmě $f'_-(1) = \infty$.

Díky spojitosti v 0 zprava a V. o lim. derivace:

2b

$$f'_+(0) \underset{\substack{\uparrow\\ \text{má-li PSS}}}{=} \lim_{x\to 0_+} f'(x) = \lim_{x\to 0_+} \frac{-1}{(x+1)^2+1} = \frac{-1}{(0+1)^2+1} = \underline{\underline{-\frac{1}{2}}}.$$

Pro zajímavost: $\lim\limits_{x\to 0_-} f'(x) = \ldots = -\frac{1}{2}$, ale $\neq f'_-(0)$.

$$f''(x) = \left(\frac{-1}{(x+1)^2+1}\right)' = -\frac{-1}{((x+1)^2+1)^2}\cdot 2(x+1) = \frac{2(x+1)}{\underbrace{((x+1)^2+1)^2}_{>0,\ x\in\mathbb{R}}}$$

4b

	$(-\infty,-1)$	$(-1,0)$	$(0,\infty)$
f''	$\ominus$	$\oplus$	$\oplus$
f	$\cap$	$\cup$	$\cup$

$\Rightarrow$ Bod $\underline{-1}$ je inflexní bod f.

$\left[\begin{array}{l} f(-1) = \operatorname{arctg}(-1) = -\frac{\pi}{4} \\ f'(-1) = -1 \end{array}\right.$

5b