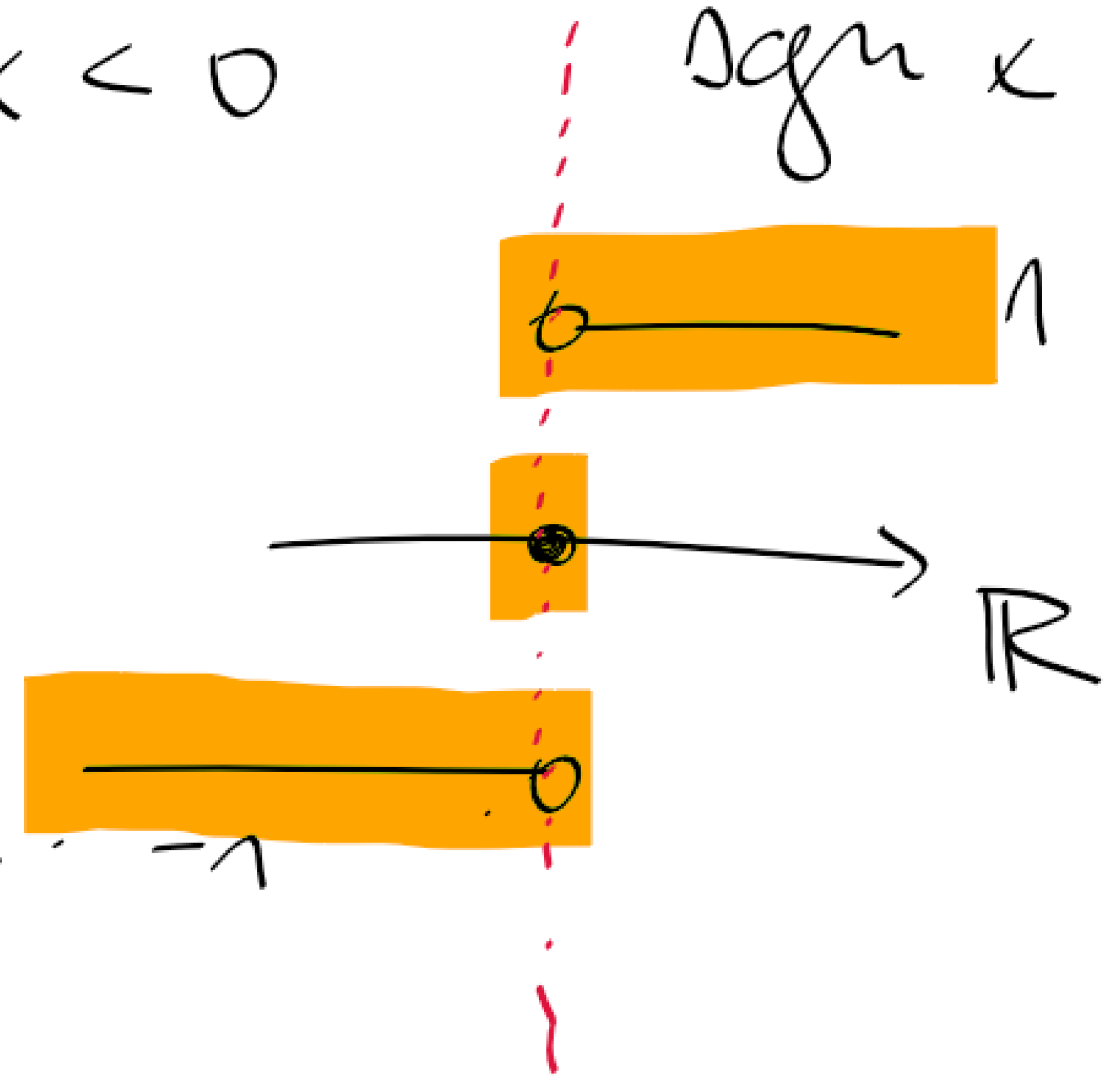
PROSEM. II 1: Funkce sgn, $|\cdot|$, související.

$$\operatorname{sgn} x = \begin{cases} 1 & \text{, pokud } x > 0 \\ 0 & \text{, } x = 0 \\ -1 & \text{, } x < 0 \end{cases}$$



Resp.:

$$\operatorname{sgn} x = \begin{cases} \frac{x}{|x|} & \text{, } x \neq 0 \\ 0 & \text{jinak} \end{cases}$$

Resp.

$$\operatorname{sgn} x = \begin{cases} \frac{|x|}{x} & \text{, } x \neq 0 \\ 0 & \text{jinak.} \end{cases}$$

Tedy také platí:

$|x| = x \cdot \operatorname{sgn} x$ resp. $\boxed{x = \operatorname{sgn} x \cdot |x|}$

Tím pádem platí třeba:

- $|\cos x| = \cos x \cdot \operatorname{sgn}(\cos x)$
- $|x^2 - 1| = (x^2 - 1) \cdot \operatorname{sgn}(x^2 - 1)$

Připomeňme: derivace složené fce:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$\big(f(g(x))\big)' = f'(\underbrace{g(x)}_{\in \mathbb{R}}) \cdot \underbrace{g'(x)}_{\in \mathbb{R}}$$

$$\big(\underbrace{\operatorname{sgn}}_{f}(\underbrace{\cos x}_{g})\big)' = \operatorname{sgn}'(\cos x) \cdot \cos' x =$$

$$= \operatorname{sgn}'(\cos x) \cdot (-\sin x),$$

pro $x \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$, $k \in \mathbb{Z}$

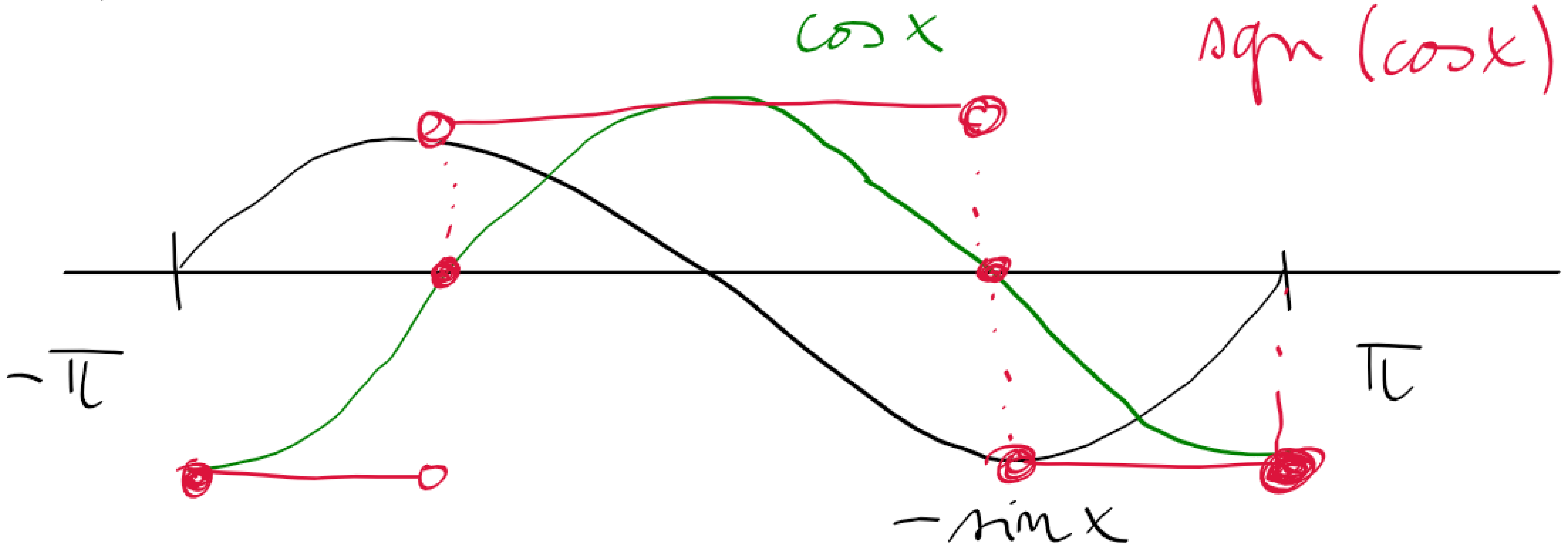
$$(|\cos x|)' = (\cos x \cdot \operatorname{sgn}(\cos x))' =$$

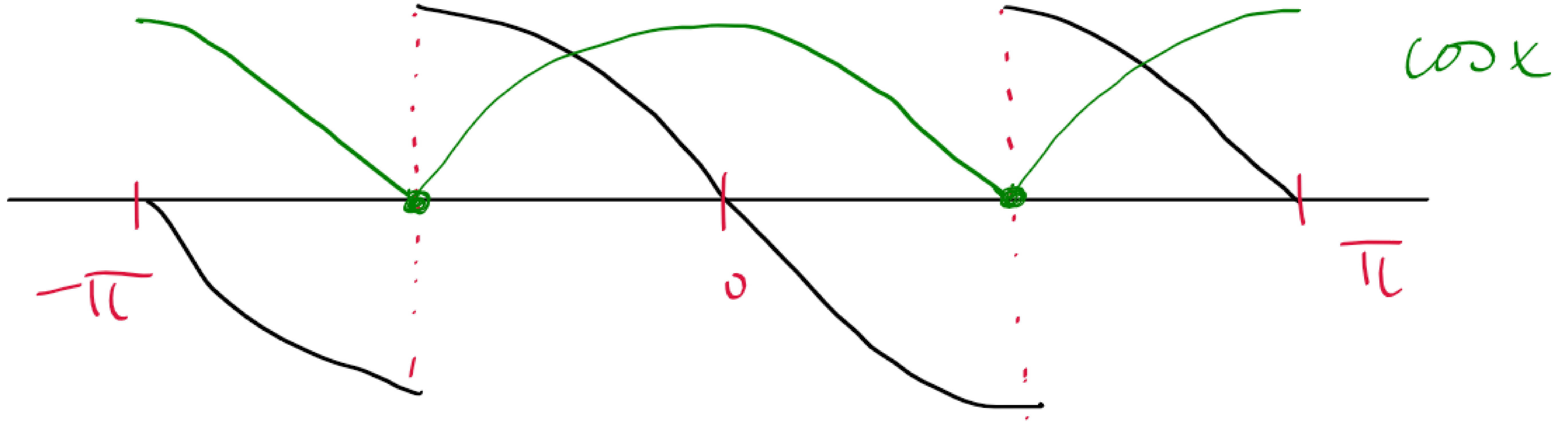
$$= (\cos x)' \cdot \operatorname{sgn}(\cos x) + \cos x \cdot \underbrace{(\operatorname{sgn}(\cos x))'}_{0},$$

$$= -\sin x \cdot \operatorname{sgn}(\cos x) + \cos x \cdot 0$$

$$x \in \bigcup_{k\in\mathbb{Z}} \left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$$

$$= -\sin x \cdot \operatorname{sgn}(\cos x) + 0$$





$$(|\cos x|)'$$

Výpočet proběhl naráz:

$$(|\cos x|)' = -\sin x \cdot \operatorname{sgn}(\cos x)$$

Nemuseli jsme rozlišovat

a) $\cos x > 0 \ldots |\cos x| = \cos x \ldots$

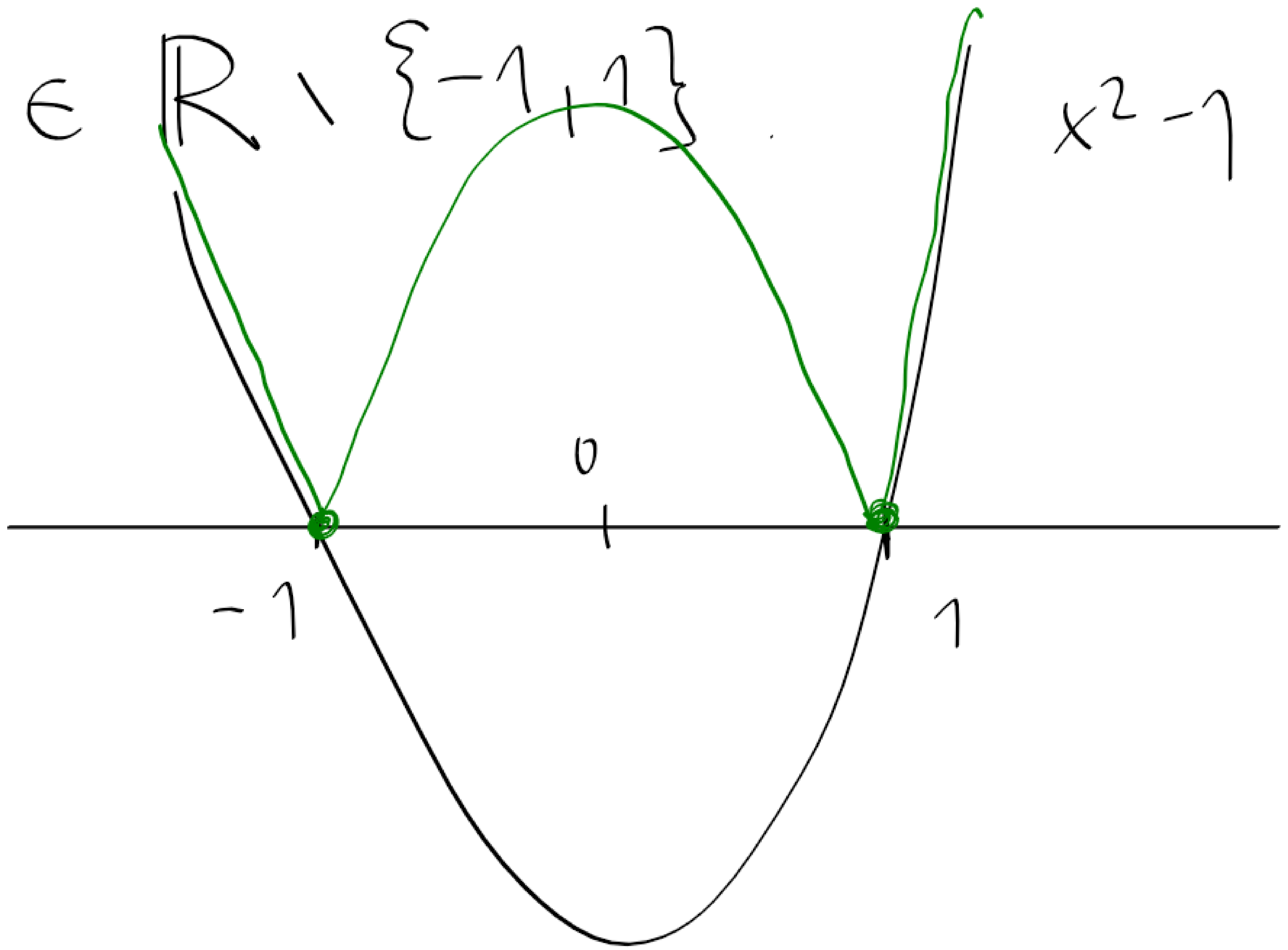
b) $\cos x < 0 \ldots |\cos x| = -\cos x \ldots$

$$(|x^2-1|)' = \Big((x^2-1)\cdot \underbrace{sgn(x^2-1)}_{\text{„konst."}}\Big)' =$$

$$= sgn(x^2-1)\cdot(x^2-1)' =$$

$$= sgn(x^2-1)\cdot(2x) = \boxed{sgn(x^2-1)\cdot 2x}$$

$$x \in \mathbb{R} \setminus \{-1,1\}$$



Složitě:

$$\begin{cases} \underline{(-\infty,-1)}: & (x^2-1)' = +2x \\ \underline{(-1,1)}: & (1-x^2)' = -2x \\ (1,\infty): & (x^2-1)' = +2x \end{cases}$$

Příklad: $\arcsin \frac{2x}{x^2+1} =: f(x)$

1) $D_f = \left\{x : \frac{2x}{x^2+1} \in D_{\arcsin}\right\} = \mathbb{R}$

$$-1 \le \frac{2x}{x^2+1} \le 1 \qquad \sqrt{x^2} = |x|$$

$$-x^2-1 \le 2x \le x^2+1 \qquad \frac{1}{\operatorname{sgn} x} = \operatorname{sgn} x, \quad x \neq 0$$

$$-x^2-2x-1 \le 0 \le x^2-2x+1$$

$$-(x+1)^2 \le 0 \le (x-1)^2 \iff x \in \mathbb{R}$$

2) f je lichá: $f(-x) = \arcsin \frac{-2x}{x^2+1} = \dots = -f(x)$

3) $\lim_{x \to \pm\infty} f(x) = \arcsin 0 = 0$.

4) Derivace: $f'(x) = \left(\arcsin \frac{2x}{x^2+1}\right)' =$

$$= \frac{1}{\sqrt{1-\left(\frac{2x}{x^2+1}\right)^2}} \cdot \frac{2\cdot(x^2+1) - 2x \cdot 2x}{(x^2+1)^2} =$$

$$= \frac{1}{\left(\frac{(x^2+1)^2 - 4x^2}{(x^2+1)^2}\right)^{\frac{1}{2}}} \cdot \frac{-2x^2+2}{(x^2+1)^2} =$$

$$= \left(\frac{(x^2+1)^2}{\underbrace{x^4-2x^2+1}_{(x^2-1)^2}}\right)^{\frac{1}{2}} \cdot \frac{-2x^2+2}{(x^2+1)^2} =$$

$$= \frac{\cancel{|x^2+1|} = x^2+1}{|x^2-1|} \cdot \frac{2\cdot(1-x^2)}{(x^2+1)^{\cancel{2}}} =$$

$$\operatorname{sgn}(x^2-1)\cdot(x^2-1) = \operatorname{sgn}(1-x^2)\cdot(1-x^2)$$

$x \in \mathbb{R} \setminus \{-1, 1\}$

$$= \frac{2\,\cancel{(1-x^2)}}{\operatorname{sgn}(1-x^2)\cdot\cancel{(1-x^2)}} \cdot \frac{1}{x^2+1} = \frac{2\operatorname{sgn}(1-x^2)}{x^2+1}.$$

$x \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$? ?

$f'(1)$, $f'(-1)$ … zvlášť

$$f''(x) = \left(\frac{2\,\mathrm{sgn}(1-x^2)}{1+x^2}\right)' =$$

$$= 2\,\mathrm{sgn}(1-x^2)\cdot\left(\frac{1}{1+x^2}\right)' =$$

$$= 2\,\mathrm{sgn}(1-x^2)\cdot\frac{-2x}{(1+x^2)^2}.$$ Dokončení ... $\underline{\omega}$.

Příklad: $f(x) = e^{-\left|\frac{x}{1-x}\right|}$

$$f'(x) = \left(e^{-\left|\frac{x}{1-x}\right|}\right)' = e^{-\left|\frac{x}{1-x}\right|}\cdot\underbrace{\left(-\left|\frac{x}{1-x}\right|\right)'}_{(*)}$$

$$(*) = \left(-\left|\frac{x}{1-x}\right|\right)' = \left(-\mathrm{sgn}\left(\frac{x}{1-x}\right)\cdot\frac{x}{1-x}\right)' =$$

$$= -\mathrm{sgn}\left(\frac{x}{1-x}\right)\cdot\left(\frac{x}{1-x}\right)' = -\mathrm{sgn}\frac{x}{1-x}\cdot\frac{1-x+x}{(1-x)^2} =$$

$$= -\mathrm{sgn}\frac{x}{1-x}\cdot\frac{1}{(1-x)^2} \qquad x\in\mathbb{R}\setminus\{0,1\}.$$

Funkce X Inversní fce

0) $e^x \times \cancel{\ln y}$

$$e^{\ln y} = y \qquad \ln e^x = x$$ OK

Platí vždy? ANO

1) $\operatorname{tg} x$ VS. $\operatorname{arctg} y$

VŽDY

$$\operatorname{tg}(\operatorname{arctg} y) = y$$

$y \in \mathbb{R}$

$\operatorname{arctg} y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

SELŽE: $x = \pi$

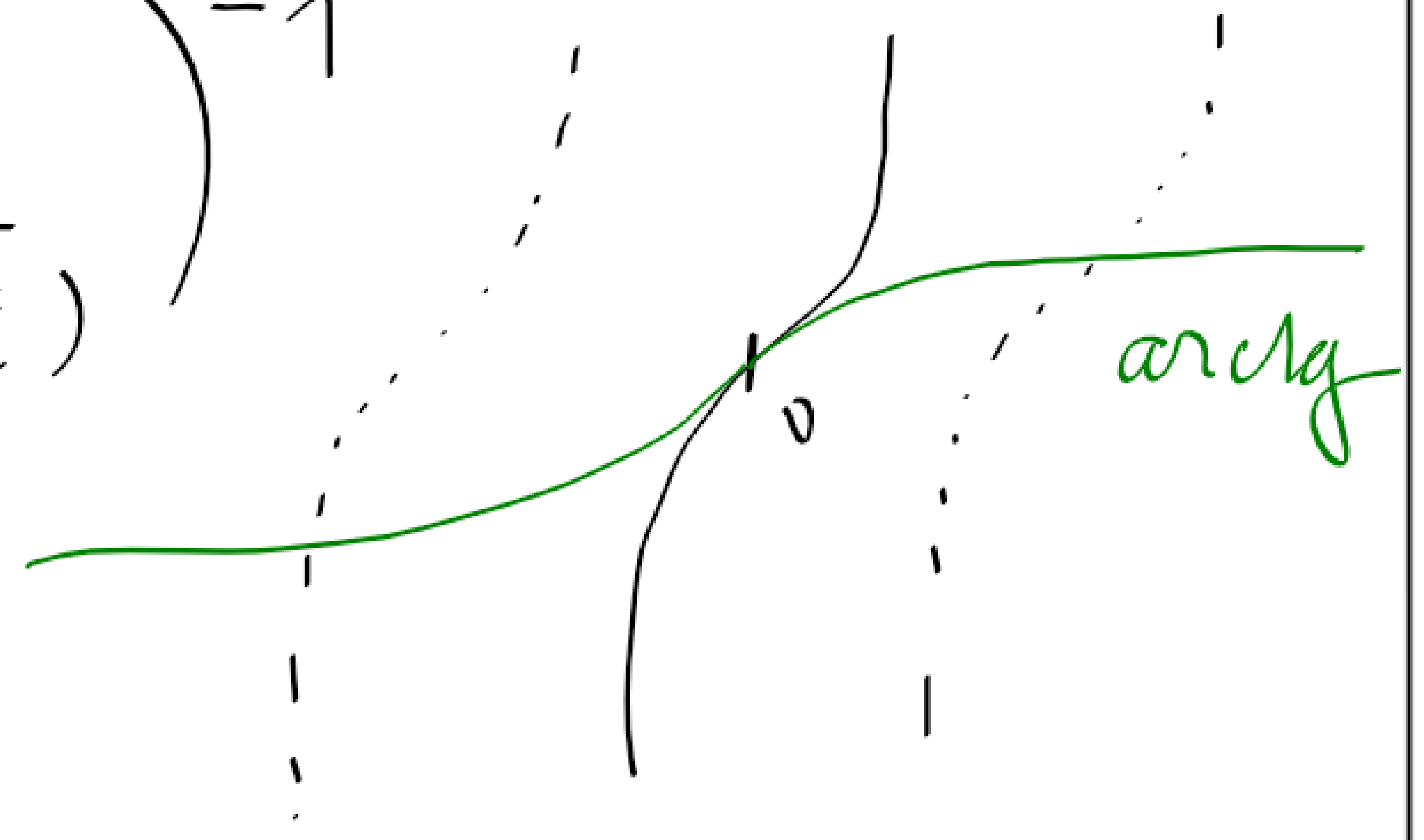
$$\operatorname{arctg}(\operatorname{tg} x) = x$$

$x \in \left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$, $k\in\mathbb{Z}$

$\operatorname{tg} x \in \mathbb{R}$

$\operatorname{arctg}(\operatorname{tg} x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\operatorname{arctg} = \left(\operatorname{tg}\big|_{(-\frac{\pi}{2},\frac{\pi}{2})}\right)^{-1}$$



$$\operatorname{arctg}(\operatorname{tg} x) = x + k\pi,$$

$$x \in \left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right),$$

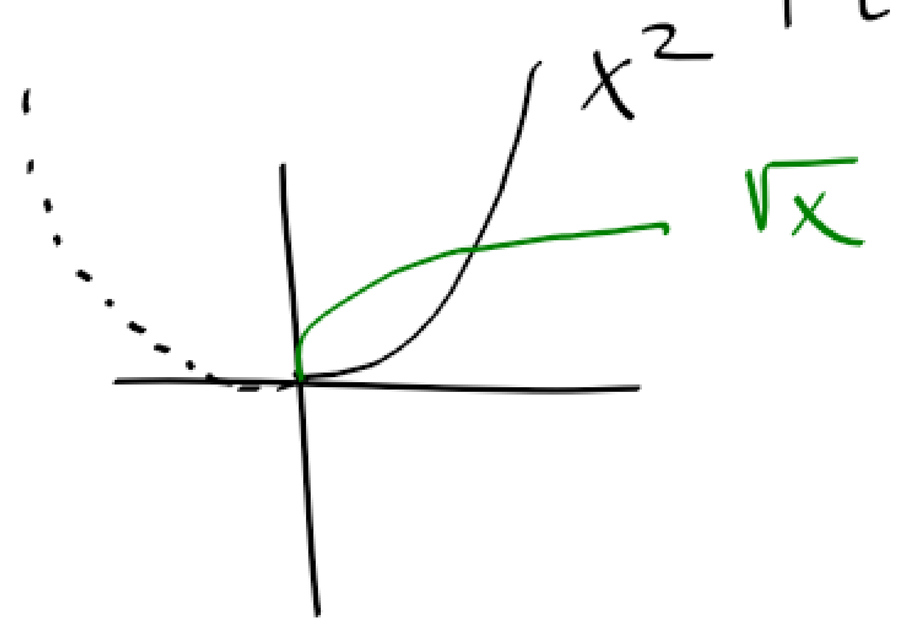
$$k \in \mathbb{Z}$$

$$\arcsin(\sin x) = x \qquad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

2) $\sqrt{x^2} = x$ obecně neplatí

$\sqrt{x^2} = |x|$, $x \in \mathbb{R}$.

$$\sqrt{\cdot} = \left((\cdot)^2\big|_{[0,\infty)}\right)^{-1}$$ KONVENCE



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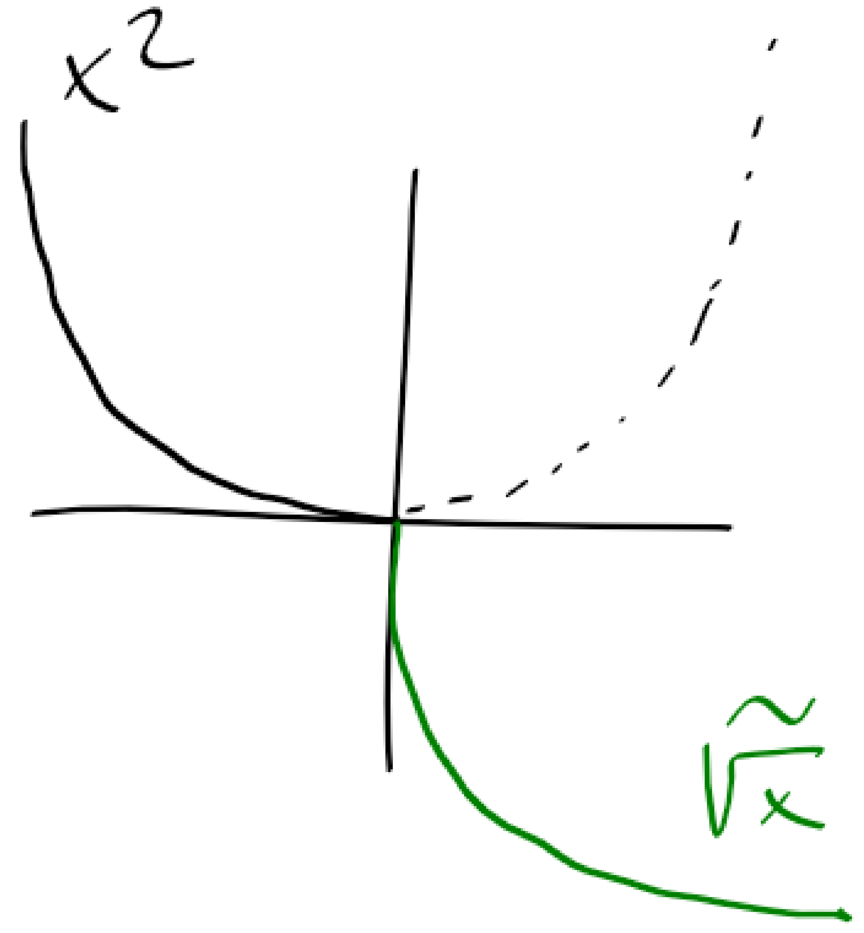
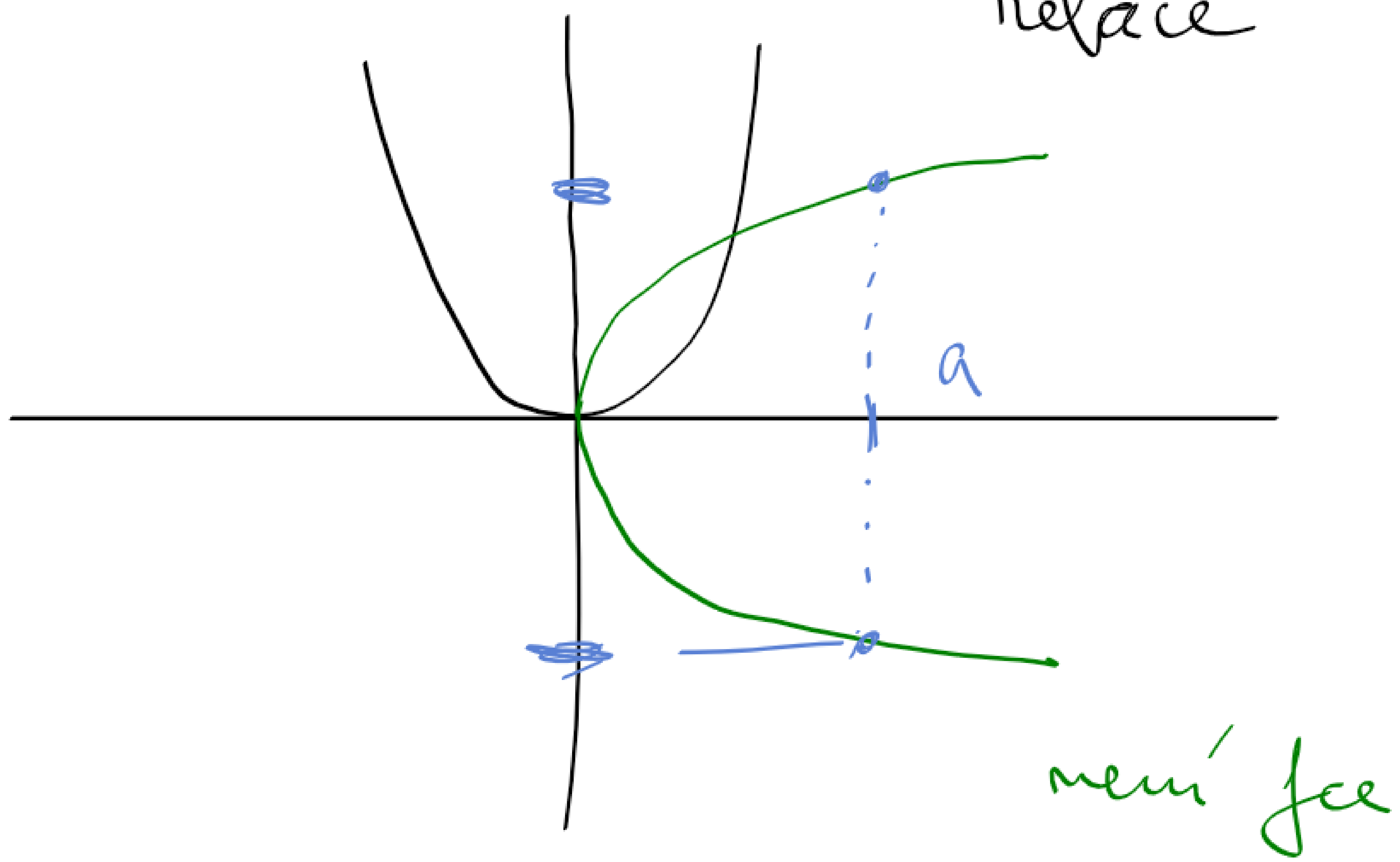
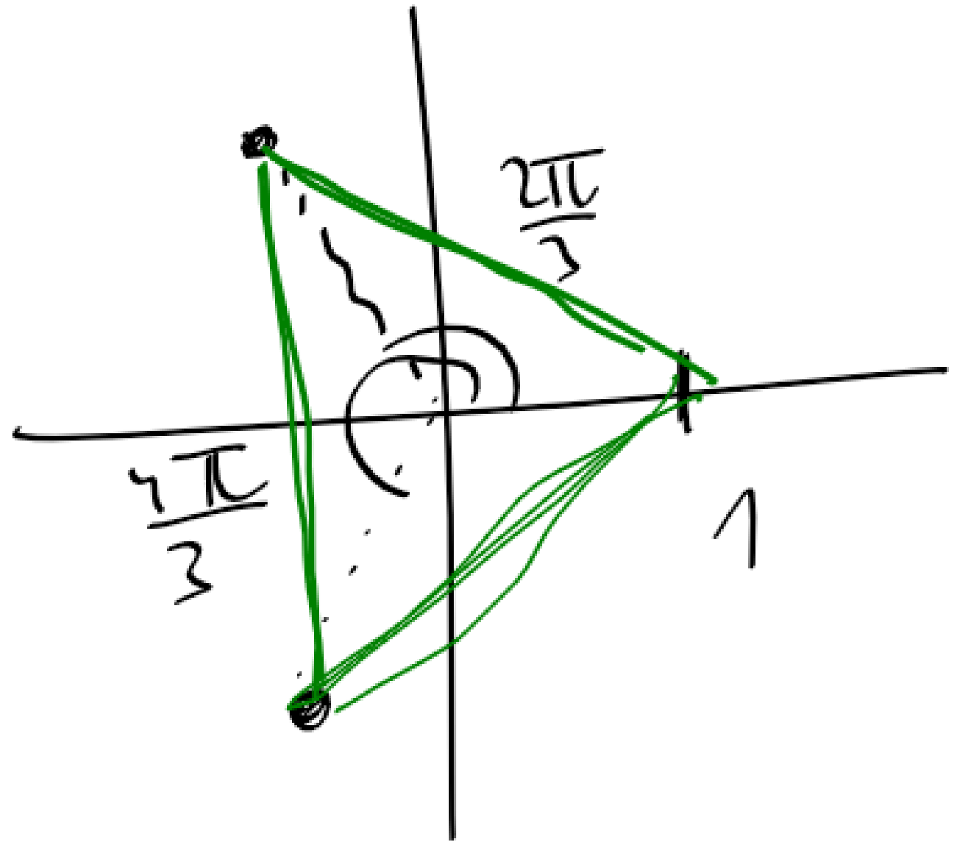
alternativní KONV. (nepoužívaná)

$$\widetilde{\sqrt{\cdot}} = \left((\cdot)^2\big|_{(-\infty,0]}\right)^{-1}$$







$\sqrt[3]{1} = z$

$z^3 = 1$

$z^3 - 1$ ... kořeny

Pozn.: • $f(x) = \sqrt[3]{x}$

Q: $D_f = ?$

a) $D_f = \mathbb{R}$ VŠ

b) $D_f = [0, \infty)$ SŠ

c) $D_f = \emptyset$