(1.) $y' = x^2 e^x \cdot y$, $y(2) = 1$. (S.Ř. $y \equiv 0$)

$$\frac{y'}{y} = x^2 e^x \qquad (y \neq 0)$$

$$\ln|y| \stackrel{c}{=} \int x^2 e^x \, dx \stackrel{\text{P.P.}}{=} x^2 e^x - \int 2x e^x \, dx \stackrel{\text{P.P.}}{=} x^2 e^x - 2\left(x e^x - \int e^x \, dx\right) =$$

$$\stackrel{c}{=} e^x \cdot (x^2 - 2x + 2) \,, \text{ tedy}$$

$$\ln|y| = e^x(x^2 - 2x + 2) + C \,, \quad C \in \mathbb{R}$$

$$|y| = e^{e^x(x^2-2x+2)} \cdot e^C$$

$$y = \pm e^C \cdot e^{e^x(x^2-2x+2)} \;; \text{ zahrneme } y \equiv 0 :$$

$$y(x) = K \cdot e^{e^x(x^2-2x+2)} \,, \quad x \in \mathbb{R} \,, \quad K \in \mathbb{R} .$$

Počáteční podmínka:

$$1 = y(2) = K \cdot e^{e^2(4-4+2)} = K \cdot e^{2e^2} \,, \text{ odkud}$$

$$K = e^{-2e^2} \,, \text{ a } \quad y(x) = e^{-2e^2} e^{e^x(x^2-2x+2)} \,, \quad x \in \mathbb{R}$$

(2.) $y' = \underbrace{\sqrt[3]{\operatorname{tg} y} \cdot \sqrt{y^4 - y^2}}_{=: g(y)}$ ... $y^4 - y^2 = y^2 (y-1)(y+1)$

$D_g = \mathbb{R} \setminus \left(\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \} \cup (-1,0) \cup (0,1) \right)$

Stacionární řešení:

$y \equiv k\pi$, $k \in \mathbb{Z}$ (včetně $k = 0$)

$y \equiv 1$, $y \equiv -1$.

Lepení: • $y \equiv 0$ ... není co lepit

• ± 1: „Srovnáme s $\frac{1}{\sqrt{y-1}}$" (resp. s $\frac{1}{\sqrt{y+1}}$):

$$\int_1^{5/4} \frac{1}{g} = \int_1^{\frac{5}{4}} \frac{dy}{\sqrt[3]{\operatorname{tg} y} \cdot \sqrt{y^4 - y^2}} = \int_1^{\frac{5}{4}} \frac{dy}{\sqrt[3]{\operatorname{tg} y} \cdot |y| \sqrt{y-1} \cdot \sqrt{y+1}}$$

LSK: $\lim\limits_{y \to 1+} \dfrac{\frac{1}{\sqrt[3]{\operatorname{tg} y} \cdot |y| \cdot \sqrt{y-1} \sqrt{y+1}}}{\frac{1}{\sqrt{y-1}}} = \lim\limits_{y \to 1+} \dfrac{\sqrt{y-1}}{\sqrt[3]{\operatorname{tg} y} \cdot |y| \cdot \sqrt{y-1} \sqrt{y+1}} =$

$= \lim\limits_{y \to 1+} \dfrac{1}{\sqrt[3]{\operatorname{tg} y} \cdot |y| \cdot \sqrt{y+1}} = \dfrac{1}{\sqrt[3]{\operatorname{tg} 1} \cdot |1| \cdot \sqrt{2}} = \dfrac{1}{\sqrt[3]{\operatorname{tg} 1} \cdot \sqrt{2}} \in (0, \infty)$

Tedy $\int_1^{\frac{5}{4}} \frac{1}{g}$ k. $\overset{\text{LSK}}{\Longleftrightarrow}$ $\int_1^{\frac{5}{4}} \frac{1}{\sqrt{y-1}} dy$ k. $\underset{\text{subst.}}{\overset{\text{triv.}}{\Longleftrightarrow}}$ $\int_0^{\frac{1}{4}} \frac{1}{\sqrt{y}} dy$ k. Platí!

Tedy $\int_1^{5/4} \frac{1}{g}$ k. a na 1 lze lepit (shora).

Analogicky: na -1 lze lepit zdola.

• $y \equiv k\pi$, $k \in \mathbb{Z} \setminus \{0\}$.

$\int_{k\pi}^{k\pi+1} \frac{1}{g} = \int_{k\pi}^{k\pi+1} \frac{dy}{\sqrt[3]{\operatorname{tg} y} \cdot \sqrt{y^4 - y^2}}$ „srovnáme s $\frac{1}{\sqrt[3]{y - k\pi}}$";

LSK: $\lim\limits_{y \to k\pi+} \dfrac{\frac{1}{g}}{\frac{1}{\sqrt[3]{y-k\pi}}} = \lim\limits_{y \to k\pi+} \dfrac{\sqrt[3]{y-k\pi}}{\sqrt[3]{\operatorname{tg} y} \cdot \sqrt{y^4 - y^2}} = \dfrac{1}{\sqrt{k^4\pi^4 - k^2\pi^2}} \cdot \lim\limits_{y \to k\pi+} \sqrt[3]{\dfrac{y-k\pi}{\operatorname{tg} y}}$

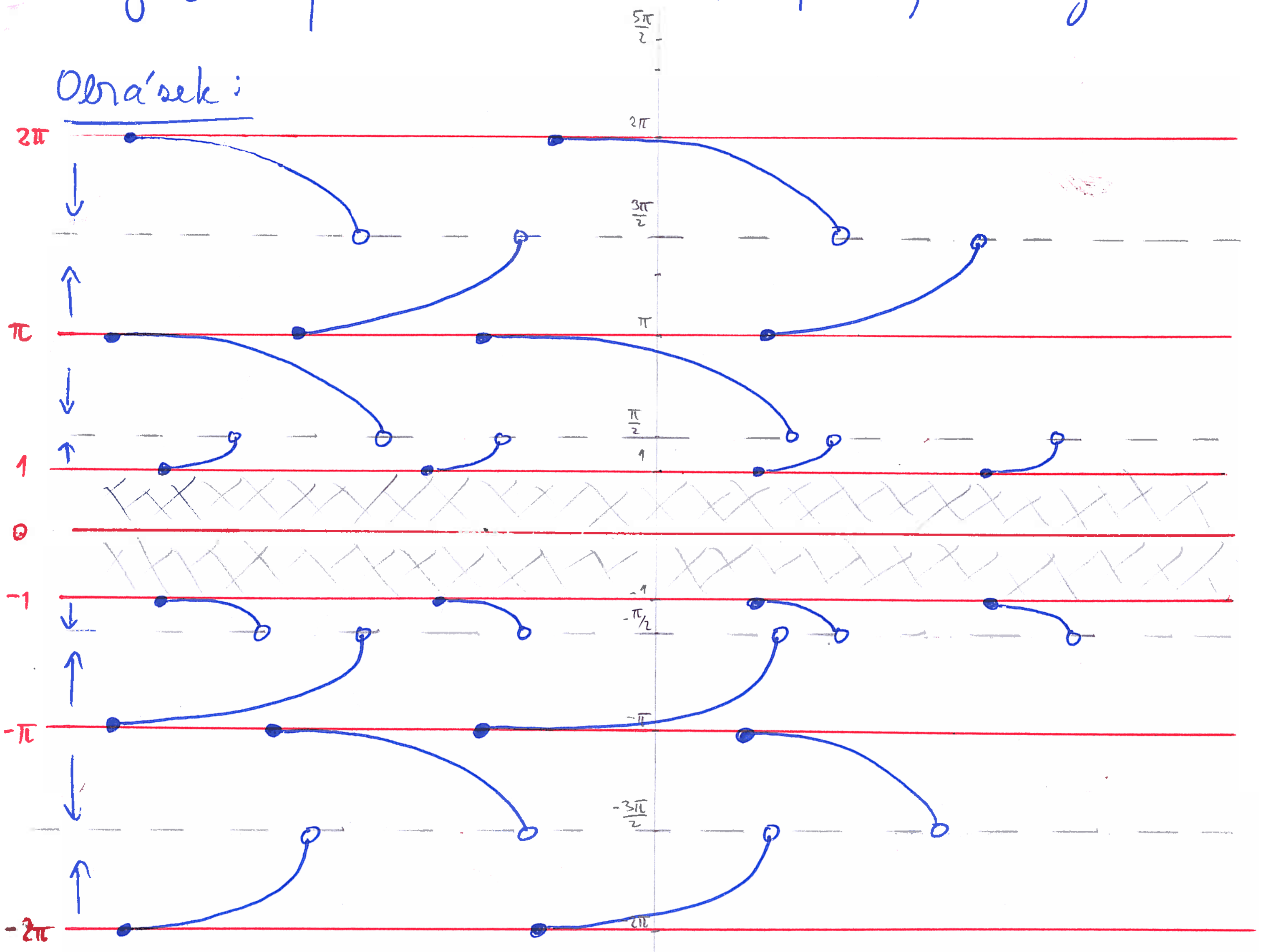
$$= \frac{1}{\sqrt{k^4\pi^4 - k^2\pi^2}} \cdot \lim_{\tilde{y}\to 0_+} \sqrt[3]{\frac{\tilde{y}}{\operatorname{tg}(\tilde{y} + k\pi)}} = \frac{1}{\sqrt{\cdots}} \cdot \lim_{\tilde{y} \to 0_+}\sqrt[3]{\frac{\tilde{y}}{\operatorname{tg}\tilde{y}}} =$$

$$= \frac{1}{\sqrt{\cdots}} \cdot \sqrt[3]{\lim_{\tilde{y}\to 0_+} \frac{\tilde{y}}{\operatorname{tg}\tilde{y}}} = \frac{1}{\sqrt{k^4\pi^4-k^2\pi^2}}\cdot\sqrt[3]{1} \in (0,\infty).$$

Tedy $\int_{k\pi}^{k\pi+1} \frac{1}{g}$ K $\iff \int_{k\pi}^{k\pi+1}\frac{dy}{\sqrt[2]{y-k\pi}}$ K. , a ten K.

Tedy lze lepit na $k\pi$ ($k\neq 0$) shora; analog. i zdola.

Obrázek:



⋮

③ $y''' + y' - 10y = 0$

CH. P.: $\lambda^3 + \lambda - 10$ ... uhodneme kořen $\lambda = 2$;

$(\lambda^3 + \lambda - 10) : (\lambda - 2) = \lambda^2 + 2\lambda + 5$.

$-(\lambda^3 - 2\lambda^2)$

$2\lambda^2 + \lambda - 10$

$-(2\lambda^2 - 4\lambda)$

$5\lambda - 10$

$\lambda^2 + 2\lambda + 5 = 0$ $\qquad \sqrt{-16} = 4i$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i$$

Tedy $\chi(\lambda) = \lambda^3 + \lambda - 10 = (\lambda - 2)(\lambda - (-1 + 2i)) \cdot (\lambda - (-1 - 2i))$.

F.S. rovnice: $\{ e^{2t}, \; e^{-t}\cos 2t, \; e^{-t}\sin 2t \}$

Obecné řešení: $y(t) = A e^{2t} + B e^{-t}\cos 2t + C e^{-t}\sin 2t$

$$y'(t) = 2A e^{2t} + B(-e^{-t}\cos 2t - 2e^{-t}\sin 2t) + C(-e^{-t}\sin 2t + 2e^{-t}\cos 2t) =$$

$$= \underbrace{2A}_{A_1} e^{2t} + \underbrace{(2C - B)}_{B_1} e^{-t}\cos 2t + \underbrace{(-2B - C)}_{C_1} e^{-t}\sin 2t$$

$-2C$ $\qquad$ $\cancel{8}-4B-2C+B$ $\qquad$ $-4C+2B+2B+C$

$$y''(t) = 2A_1 e^{2t} + (2C_1 - B_1) e^{-t}\cos 2t + (-2B_1 - C_1) e^{-t}\sin 2t$$

$$= \underbrace{4A}_{A_2} e^{2t} + \underbrace{(-4C - 3B)}_{B_2} e^{-t}\cos 2t + (\underbrace{4B - 3C}_{C_2}) e^{-t}\sin 2t$$

$\cancel{y'''(t)} = 2A_2 e^{2t} + (2C_2 - B_2) e^{-t}\cos 2t + (-2B_2 - C_2) e^{-t}\sin 2t$

~~$= 8A e^{2t} + (3B - 2C) e^{-t}\cos 2t + (11C + 6B) e^{-t}\sin 2t$~~

Počáteční podmínky:

$$1 = y(0) = A \cdot 1 + B \cdot 1 + C \cdot 0$$

$$1 = y'(0) = 2A + (-B) + 2C$$

$$0 = y''(0) = 4A - 3B - 4C$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & -1 & 2 & 1 \\ 4 & -3 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -3 & 2 & -1 \\ 0 & -7 & -4 & -4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -3 & 2 & -1 \\ 0 & -13 & 0 & -6 \end{array}\right)$$

$$\underline{\underline{B = \frac{6}{13}}}$$

$$-3B + 2C = -1$$

$$-\frac{18}{13} + 2C = -1$$

$$2C = \frac{5}{13}$$

$$\underline{\underline{C = \frac{5}{26}}}$$

$$A = 1 - \frac{6}{13} =$$

$$\underline{\underline{A = \frac{7}{13}}}$$

Tedy řešení úlohy s počáteční podmínkou je fce:

$$y(t) = \frac{7}{13} e^{2t} + \frac{6}{13} e^{-t} \cos 2t + \frac{5}{26} e^{-t} \sin 2t,$$

$$t \in \mathbb{R}.$$