① $y' = (1+y^2)\,\mathrm{tg}\,x \quad , \quad y(0) = 1 .$ STAC. Ř. NENÍ

$$x \in D_{\mathrm{tg}} = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \; : \; k \in \mathbb{Z} \right\}$$

$$\frac{y'}{1+y^2} = \mathrm{tg}\,x$$

$$\mathrm{arctg}\, y \overset{c}{=} \int \frac{\sin x}{\cos x}\,dx = \left| \begin{matrix} z = \cos x \\ dz = -\sin x\,dx \end{matrix} \right| = -\int \frac{dz}{z} \overset{c}{=}$$

$$\overset{c}{=} -\ln|z| = -\ln|\cos x| .$$ Tedy:

$$\boxed{\mathrm{arctg}\, y = -\ln|\cos x| + C} \quad , \quad C \in \mathbb{R}$$

$$y(x) = \mathrm{tg}\left(-\ln|\cos x| + C\right)$$

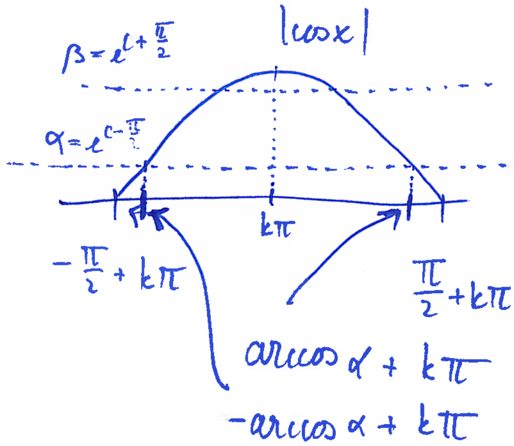
$$\Longrightarrow -\ln|\cos x| + C \in H_{\mathrm{arctg}} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) ,$$ tj.

$$-\frac{\pi}{2} < C - \ln|\cos x| < \frac{\pi}{2}$$

$$-\frac{\pi}{2} - C < -\ln|\cos x| < \frac{\pi}{2} - C$$

$$\frac{\pi}{2} + C > \ln|\cos x| > C - \frac{\pi}{2}$$

$$e^{\frac{\pi}{2}+C} > |\cos x| > e^{C-\frac{\pi}{2}}$$



- $x \in \emptyset$ pro $e^{C-\frac{\pi}{2}} \geq 1$, tj. $C - \frac{\pi}{2} \geq 0$, tj. $C \geq \frac{\pi}{2}$
- $C \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$: $\underbrace{e^{\frac{\pi}{2}+C}}_{\beta} > 1$, takže $x \in \left(-\arccos\big(\overbrace{e^{C-\frac{\pi}{2}}}^{\alpha}\big) + k\pi, \arccos\big(\underbrace{e^{C-\frac{\pi}{2}}}_{\alpha}\big) + k\pi\right)$
- $C \leq -\frac{\pi}{2}$: $x \in \left(-\arccos\left(e^{C-\frac{\pi}{2}}\right) + k\pi, -\arccos\left(e^{C+\frac{\pi}{2}}\right) + k\pi\right)$,

$$x \in \left(\arccos\left(e^{c+\frac{\pi}{2}}\right) + k\pi \,,\, \arccos\left(e^{c-\frac{\pi}{2}}\right) + k\pi \right)$$

POČÁTEČNÍ PODMÍNKA:

$$1 = y(\underline{\underline{0}}) = \operatorname{tg}\left(-\ln|\cos 0| + c\right) =$$

$$= \operatorname{tg}\left(-\ln 1 + c\right) = \operatorname{tg} c$$

Tedy $c \in \left\{ \frac{\pi}{4} + k\pi \,;\, k \in \mathbb{Z} \right\}$ (tg je π-periodická!)

ale pouze pro $c = \frac{\pi}{4}$ je (podle rozboru výše)

$\underline{\underline{0}} \in D_y$, což musí, jinak P.P. $y(\underline{0}) = 1$ nelze splnit.

Tedy hledané řešení je:

$$y(x) = \operatorname{tg}\left(\frac{\pi}{4} - \ln|\cos x|\right),$$

$$x \in \left(-\arccos\left(e^{-\frac{\pi}{4}}\right) \,,\, \arccos\left(e^{-\frac{\pi}{4}}\right)\right)$$

② $y' = \sqrt{|y^3-1|}\cdot \operatorname{arctg} y =: g(y) \qquad y\in\mathbb{R}$,

STAC. ŘEŠENÍ: $\quad y\equiv 1 \quad$ a $\quad y\equiv 0$.

MONOTONIE: $\quad y'\geq 0 \iff \operatorname{arctg} y \geq 0 \iff y\geq 0$

$$\begin{array}{ccccc} & - & & + & & + \\ \hline & & 0 & & 1 & \end{array}$$

Lepení: na 0: $\displaystyle\int_0^{\frac12} \frac{dy}{\sqrt{|y^3-1|}\cdot\operatorname{arctg} y}$ ... srovnej s $\frac{1}{y}$:

$$\lim_{y\to 0}\frac{\frac{1}{g(y)}}{\frac{1}{y}} = \lim_{y\to 0}\frac{y}{\sqrt{|y^3-1|}\cdot\operatorname{arctg} y} = \lim_{y\to 0}\frac{1}{\sqrt{|y^3-1|}}\cdot\lim_{y\to 0}\frac{y}{\operatorname{arctg} y}$$

$$\underset{\left(\frac{0}{0}\right)}{\overset{\text{l'H}}{=}} \frac{1}{\sqrt{|0^3-1|}}\cdot\lim_{y\to 0}\frac{1}{\frac{1}{1+y^2}} = 1\cdot\frac{1}{\frac{1}{1+0^2}} = 1\in(0,\infty)$$

Tedy $\displaystyle\int_0^{\frac12}\frac{1}{g}$ K. $\iff \displaystyle\int_0^{\frac12}\frac{dy}{y}$ K., ale ten D. $\Rightarrow$ nelze lepit!
Podobně zdola.

na 1: $\quad y^3-1 = (y-1)\cdot(y^2+y+1)$ ... srovnej s $\frac{1}{\sqrt{y-1}}$

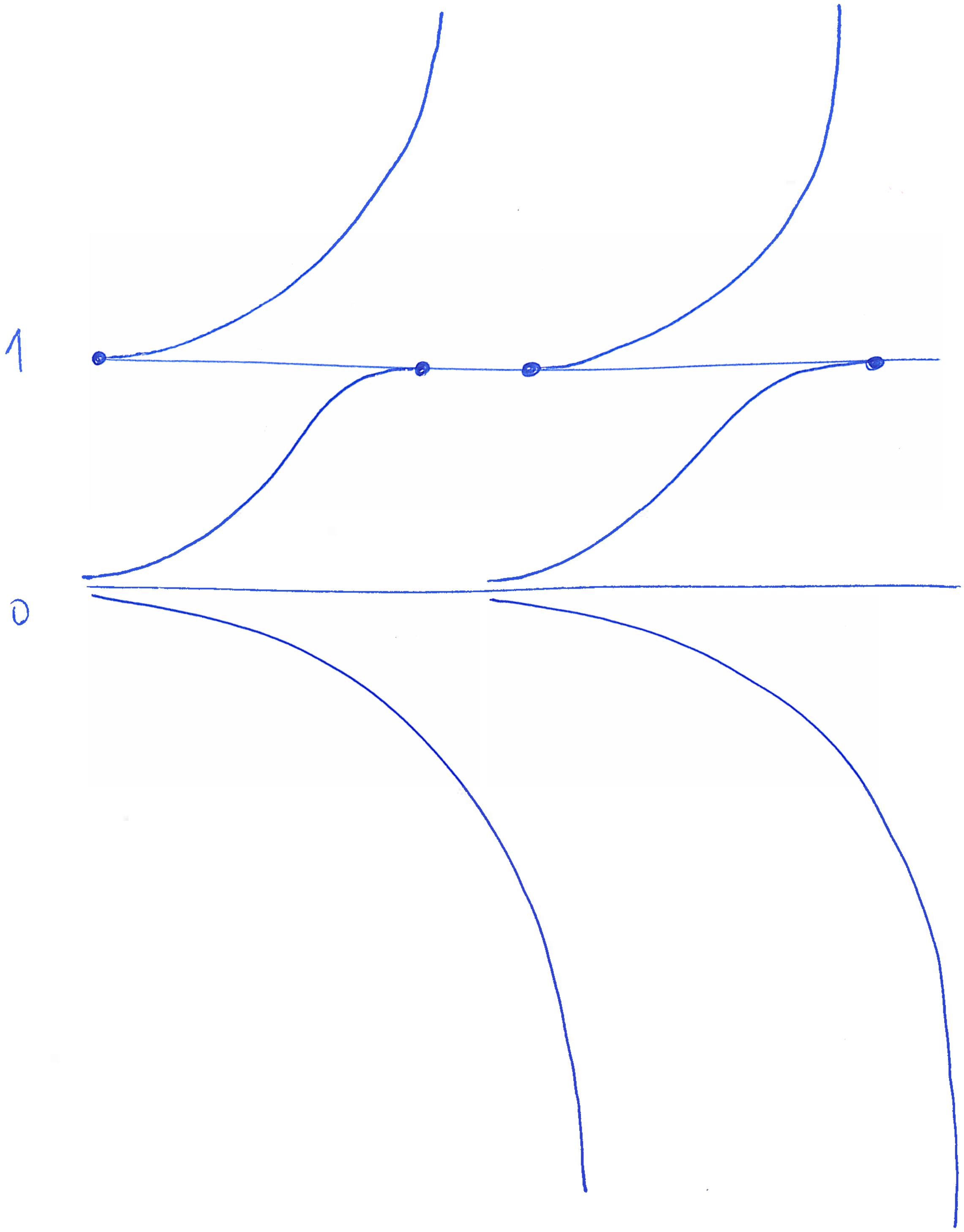
$$\lim_{y\to 1_+}\frac{\frac{1}{g(y)}}{\frac{1}{\sqrt{y-1}}} = \lim_{y\to 1}\frac{1}{\operatorname{arctg} y}\cdot\lim\frac{1}{\sqrt{|y^2+y+1|}}\cdot\lim\frac{\frac{1}{\sqrt{|y-1|}}}{\frac{1}{\sqrt{y-1}}} =$$

$$= \frac{\pi}{4}\cdot\frac{1}{\sqrt{3}}\cdot 1 \in (0,\infty),$$

tedy $\displaystyle\int_1^2\frac{1}{g}$ K. $\iff \displaystyle\int_1^2\frac{dy}{\sqrt{y-1}}$ K., ale ten K. $\Rightarrow$ lze lepit

analogicky zdola.

(Svislé asymptoty ANO: $\displaystyle\int_2^{\infty}\frac{1}{g}$ K, $\displaystyle\int_{-\infty}^{-1}\frac{1}{g}$ K. – snadno / srovnej s $\frac{1}{|y|^{3/2}}$.



1
0

(3.) • $y' + xy = e^{-\frac{1}{2}x^2}$

$p(x) = x$, $P(x) = \frac{x^2}{2}$, I.F.: $e^{\frac{x^2}{2}}$

$e^{\frac{x^2}{2}} \cdot y' + x e^{\frac{x^2}{2}} y = 1$, tj. $\left(e^{\frac{x^2}{2}} y\right)' = 1$

$e^{\frac{x^2}{2}} y = x + C$, $C \in \mathbb{R}$

$y(x) = e^{-\frac{x^2}{2}}(x + C)$, $x \in \mathbb{R}$.

• $y''' + 3y'' + 3y' + y = 0$ ... Char. polynom:

$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3$, tedy

$\lambda = -1$ je 3-násobný kořen.

Tedy F.S. $:= \{e^{-t}, te^{-t}, t^2 e^{-t}\}$ a

všechna max. řešení (jsou def. na $\mathbb{R}$) jsou

tvaru

$y(x) = C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t}$.