Parciální derivace, totální diferenciál, derivace ve směru, gradient a tečná rovina

1. Určete parciální derivace následujících funkci:
   (a) $f(x,y) = (2x-3y)^4$,
   (b) $f(x,y) = y^{x^2+3}$,
   (c) $f(x,y) = \frac{y}{\sqrt{x^2-y^2}}$,
   (d) $f(x,y,z) = (x^y)^z$,
2. Určete totální diferenciál následujících funkcí:
   (a) $f(x,y) = \frac{y}{x} - \frac{x}{y}$,
   (b) $f(x,y) = \mathrm{arctg}\frac{y}{x}$,
3. Určete přibližně hodnoty:
   (a) $\ln(\sqrt{9.03} - \sqrt{0.99} - 1)$,
   (b) $0.98^{3.04}$,
4. Určete tečnou rovinu k funkci f v bodě $A = [x_0, y_0, ?]$:
   (a) $f(x,y) = 2x^2 - 4y^2$, $A = [2, 1, ?]$,
   (b) $f(x,y) = 4\sqrt{x^2+y^2}$, $A = [3, 4, ?]$,
   (c) $f(x,y) = \frac{\mathrm{arcsin}y}{x}$, $A = [\frac{1}{2}, \frac{\sqrt{2}}{2}, ?]$,
   (d) $f(x,y) = x^2\cos\frac{1}{y}$, $A = [1, \frac{2}{\pi}, ?]$,
5. Najděte rovnici tečné roviny k funkci f, která je rovnoběžná s rovinou ρ.
   (a) $f(x,y) = x^2 + xy - y^2 + x + 3$, $\rho : 5x - 3y - z = 0$,
   (b) $f(x,y) = 2x^2 - y^2$, $\rho : 8x - 6y - z - 15 = 0$,
   (c) $f(x,y) = \ln(x^2 + 2y^2)$, $\rho : 2x - z + 5 = 0$,
6. Určete gradient funkce $f(x,y,z) = x^y + yz$.
7. Určete derivaci funkce $f(x,y) = 2x^4 + xy + y^3$ ve směru $\vec{s} = (3,4)$ v bodě $A = [1,2]$ (z definice i pomocí gradientu).

8. Určete derivaci funkce $f(x,y,z) = x^2 + 2y^2 - z^2$ ve směru $\vec{AB}$ v bodě $A = [-3,2,4]$, kde $B = [-2,4,2]$.

Řešení:

1. (a) $f_x = 8(2x-3y)^3, \quad f_y = -12(2x-3y)^3,$

   (b) $f_x = y^{x^2+3} 2x \ln y, \quad f_y = (x^2+3)y^{x^2+2},$

   (c) $f_x = \frac{-xy}{(x^2-y^2)^{\frac{3}{2}}}, \quad f_y = \frac{y^2}{(x^2-y^2)^{\frac{3}{2}}},$

   (d) $f_x = yz \cdot x^{yz-1}, \quad f_y = z\,(x^y)^{z-1} x^y \ln z = x^{yz} z \ln x, \quad f_z = x^{yz} y \ln x,$

2. (a) $df = f_x dx + f_y dy = \left(-\frac{y}{x^2} - \frac{1}{y}\right) dx + \left(\frac{1}{x} + \frac{x}{y^2}\right) dy,$

   (b) $df = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} dx + \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} dy = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy,$

3. (a)

$$f(x,y) = \ln(\sqrt{x} - \sqrt{y} - 1) \Rightarrow df = \frac{1}{(2\sqrt{x})(\sqrt{x}-\sqrt{y}-1)} dx - \frac{1}{(2\sqrt{y})(\sqrt{x}-\sqrt{y}-1)} dy$$

$$= \left| \begin{matrix} dx = 0.03 \\ dy = -0.01 \\ [x,y] = [9,1] \end{matrix} \right| \frac{1}{(2\sqrt{9})(\sqrt{9}-\sqrt{1}-1)} \cdot 0.03 - \frac{1}{(2\sqrt{1})(\sqrt{9}-\sqrt{1}-1)} \cdot (-0.01)$$

$$= \frac{0.03}{6} + \frac{0.01}{2} = 0.01 \Rightarrow \ln(\sqrt{9.03} - \sqrt{0.99} - 1) \approx f(9,1) + df = 0 + 0.01 = 0.01.$$

   (b)

$$f(x,y) = x^y \Rightarrow df = yx^{y-1} dx + x^y \ln x dy = \left| \begin{matrix} dx = -0.02 \\ dy = 0.04 \\ [x,y] = [1,3] \end{matrix} \right| 3 \cdot (-0.02) + 0 \cdot (0.04) = -0.06$$

$$0.98^{3.04} \approx f(1,3) + df = 1 - 0.06 = 0.96.$$

4. Tečná rovine:

$$z(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

   (a)

$$f_x = 4x, \quad f_y = -8y \Rightarrow z = f(2,1) + 8(x-2) - 8(y-1) = 4 + 8x - 8y - 8$$

$$z = 8x - 8y - 4.$$

(b)

$$f_x = \frac{4x}{\sqrt{x^2+y^2}}, \quad f_y = \frac{4y}{\sqrt{x^2+y^2}}, \Rightarrow z = f(3,4) + f_x(3,4)(x-3) + f_y(3,4)(y-4)$$
$$= 20 + \frac{12}{5}(x-3) + \frac{16}{5}(y-4) = \frac{100-36-64}{5} + \frac{12}{5}x + \frac{16}{5}y \Rightarrow$$
$$5z = 12x + 16y.$$

(c)

$$f_x = \frac{-\text{arcsin}y}{x^2}, \; f_y = \frac{1}{x\sqrt{1-y^2}} \Rightarrow z = f(\frac{1}{2}, \frac{\sqrt{2}}{2}) + f_x(\frac{1}{2}, \frac{\sqrt{2}}{2})(x - \frac{1}{2}) + f_y(\frac{1}{2}, \frac{\sqrt{2}}{2})(y - \frac{\sqrt{2}}{2})$$
$$= \frac{\frac{\pi}{4}}{\frac{1}{2}} - \frac{\frac{\pi}{4}}{\frac{1}{4}}(x - \frac{1}{2}) + \frac{1}{\frac{1}{2}\sqrt{1-\frac{1}{2}}}(y - \frac{\sqrt{2}}{2}) = \pi - 2 - \pi x + 2\sqrt{2}y$$
$$z = \pi - 2 - \pi x + 2\sqrt{2}y.$$

(d)

$$f_x = 2x\cos\frac{1}{y}, \; f_y = \frac{x^2}{y^2}\sin\frac{1}{y} \Rightarrow z = f(1, \frac{2}{\pi}) + f_x(1, \frac{2}{\pi})(x-1) + f_y(1, \frac{2}{\pi})(y - \frac{2}{\pi})$$
$$= 0 + 0(x-1) + \frac{\pi^2}{4}(y - \frac{2}{\pi})$$
$$z = \frac{\pi^2}{4}y - \frac{\pi}{2}.$$

5. (a)

$$f_x = 2x + y + 1 = 5, \; f_y = x - 2y = -3 \Rightarrow [x_0, y_0] = [1, 2] \Rightarrow$$
$$z = f(1,2) + f_x(x-1) + f_y(y-2) = 3 + 5(x-1) - 3(y-2) \Rightarrow$$
$$z = 4 + 5x - 3y.$$

(b)

$$f_x = 4x = 8, \; f_y = -2y = -6 \Rightarrow [x_0, y_0] = [2, 3] \Rightarrow$$
$$z = f(2,3) + f_x(x-2) + f_y(y-3) = -1 + 8(x-2) - 6(y-3) \Rightarrow$$
$$z = 1 + 8x - 6y.$$

(c)

$$f_x = \frac{2x}{x^2+2y^2} = 2, \; f_y = \frac{4y}{x^2+2y^2} = 0 \Rightarrow [x_0, y_0] = [1, 0] \Rightarrow$$
$$z = f(1,0) + f_x(x-1) + f_y y = 0 + 2(x-1) \Rightarrow$$
$$z = -2 + 2x.$$

6.

$$\mathrm{grad} f(x,y,z) = \nabla f = (f_x, f_y, f_z) = (yx^{y-1}, x^y \ln x + z, y).$$

7.

$$\begin{aligned}
\frac{\partial f}{\partial \vec{s}} &= \lim_{h\to 0} \frac{f([x,y]+h\vec{s}) - f(x,y)}{h|\vec{s}|} = \lim_{h\to 0} \frac{f(x+3h, y+4h) - f(x,y)}{5h} \\
&= \lim_{h\to 0} \frac{2(x+3h)^4 + (x+3h)(y+4h) + (y+4h)^3 - [2x^4 + xy + y^3]}{5h} \\
&= \lim_{h\to 0} \frac{2(x^4 + 12x^3h + o(h)) + xy + 4xh + 3yh + o(h) + y^3 + 12y^2h + o(h) - [2x^4 + xy + y^3]}{5h} \\
&= \lim_{h\to 0} \frac{(24x^3 + 4x + 3y + 12y^2)h + o(h)}{5h} = \frac{24x^3 + 4x + 3y + 12y^2}{5} \\
\frac{\partial f}{\partial \vec{s}} &= \frac{\nabla f \cdot \vec{s}}{|\vec{s}|} = \frac{(8x^3 + y, x + 3y^2)\cdot(3,4)}{5} = \frac{24x^3 + 4x + 3y + 12y^2}{5} \\
\frac{\partial f}{\partial \vec{s}}(A) &= \frac{24 + 4 + 6 + 48}{5} = \frac{82}{5}.
\end{aligned}$$

8.

$$\begin{aligned}
\frac{\partial f}{\partial \vec{AB}} &= \frac{\nabla f \cdot \vec{(AB)}}{|\vec{(AB)}|} = \frac{(2x, 4y, -2z)\cdot(1,2,-2)}{3} = \frac{2x + 8y + 4z}{3} \Rightarrow \\
\frac{\partial f}{\partial \vec{AB}}(A) &= \frac{-6 + 16 + 16}{3} = \frac{26}{3}.
\end{aligned}$$