Appendix

A.1 Some properties of Sobolev spaces

Lemma 1.1 *Let $\Omega \subseteq \mathbb{R}^d$ be an open set. Let $a_0 \geq 0$ be given and let $G \in C^1(\mathbb{R})$. We assume that there are constants $M, k = k(a_0)$, such that*

$$
\begin{aligned}
|G'(\xi)| &\leq M\,, \\
|G(\xi - a_0)| &\leq k|\xi|\,.
\end{aligned}
\tag{1.2}
$$

Let $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, and denote $v \equiv u - a_0$. Then $G \circ v$ belongs to $W^{1,p}(\Omega)$ and

$$\nabla(G \circ v) = G'(v)\nabla v \qquad \text{a.e. in } \Omega\,. \tag{1.3}$$

Proof : The following proof is only a slight modification of a similar result by Kinderlehrer and Stampacchia [1980]. Since $u \in W^{1,p}(\Omega)$, there exists a sequence $u^n \in C^1(\Omega)$ such that $u^n \to u$ in $W^{1,p}(\Omega)$ and almost everywhere in Ω. Let us denote $v^n \equiv u^n - a_0$, then obviously $\nabla v^n \to \nabla v$ in $L^p(\Omega)^d$ and $v^n \to v$ almost everywhere in Ω (but in general not $v^n \to v$ in $L^p(\Omega)$). Further we have $G \circ v^n \in C^1(\Omega)$ and due to $(1.2)_2$, $G \circ v^n$, $G \circ v \in L^p(\Omega)$. Now, from

$$|G(v^n) - G(v)| \leq M\,|v^n - v| \leq |u^n - u|$$

we obtain that

$$G \circ v^n \to G \circ v \qquad \text{in } L^p(\Omega)\,.$$

We will show that also

$$\frac{\partial}{\partial x_i} G(v^n) = G'(v^n)\frac{\partial v^n}{\partial x_i} \to G'(v)\frac{\partial v}{\partial x_i} \qquad \text{in } L^p(\Omega)\,. \tag{1.4}$$

Indeed,

$$\begin{aligned} G'(v^n)\frac{\partial v^n}{\partial x_i} &- G'(v)\frac{\partial v}{\partial x_i} \\ &= G'(v^n)\left(\frac{\partial v^n}{\partial x_i} - \frac{\partial v}{\partial x_i}\right) + \left(G'(v^n) - G'(v)\right)\frac{\partial v}{\partial x_i} \\ &\equiv A_n + B_n\,. \end{aligned}$$

Now, obviously, A_n converges to zero in $L^p(\Omega)$ and B_n converges to zero almost everywhere in Ω. Since

$$|B_n|^p \leq (2M)^p \left|\frac{\partial v}{\partial x_i}\right|^p,$$

we conclude from the Lebesgue dominated convergence theorem that B_n converges to zero in $L^p(\Omega)$. Since the derivatives of $G \circ v$ in the sense of distributions are the limits of $\frac{\partial}{\partial x_i}\big(G \circ v^n\big)$ in $L^p(\Omega)$, we obtain immediately (1.3). ∎

Remark 1.5 Note that if $v \in W^{1,p}(\Omega)$ is such that $v : \Omega' \to (\alpha, \beta)$, for some $\Omega' \subset \Omega$, and $G \in C^1(\alpha, \beta)$, then (1.3) holds in Ω'.

Lemma 1.6 *Let $\Omega \subseteq \mathbb{R}^d$ be an open set and let $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, and again set $v \equiv u - a_0$ for given $a_0 \geq 0$. Then for all $1 \leq i \leq d$ we have*

$$\frac{\partial v}{\partial x_i} = 0 \qquad \text{a.e. in } E = \{x \in \Omega; v(x) = 0\}\,.$$

PROOF : see for example KINDERLEHRER AND STAMPACCHIA [1980, Lemma A.4, page 53]. ∎

Note that this lemma also holds for $E = \{x \in \Omega; v(x) = a\}$, $a \in \mathbb{R}$ fixed constant.

Now we are able to give a slight generalization of Lemma 1.1.

Lemma 1.7 *Let $\Omega \subseteq \mathbb{R}^d$ be an open set. For $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, and a given constant $a_0 \geq 0$ put $v \equiv u - a_0$. Then $v^+ \equiv \max(v, 0)$ belongs to $W^{1,p}(\Omega)$ and*

$$\nabla v^+ = H(v)\nabla v \qquad \text{a.e. in } \Omega\,, \tag{1.8}$$

where $H(\xi) = 1$ if $\xi > 0$, $H(\xi) = 0$ if $\xi \leq 0$. In (1.8) the convention that both sides are zero on the set $\{x \in \Omega; v(x) = 0\}$ is used. Moreover, the mapping $u \mapsto v^+$ is continuous in $W^{1,p}(\Omega)$.

PROOF : The first part follows immediately from Lemma 1.1 and Remark 1.5 with $\Omega' = \Omega \setminus \{x \in \Omega; v(x) = 0\}$. Lemma 1.6 justifies the convention used in (1.8). It remains to show the continuity of the mapping $u \mapsto v^+$. Put $F(\xi) \equiv \max(\xi, 0)$. From (1.8) and $|F(\xi - a_0)| \leq |\xi|$ we get

$$\|v^+\|_{W^{1,p}(\Omega)} \leq \|u\|_{W^{1,p}(\Omega)} , \tag{1.9}$$

which is the boundedness of the (nonlinear) mapping $u \mapsto v^+$. From (1.9) follows that for a given sequence $u_n \to u$ in $W^{1,p}(\Omega)$ we can extract from $v_n^+ \equiv F \circ (u_n - a_0)$ a subsequence still denoted $\{v_n^+\}$ converging weakly in $W^{1,p}(\Omega)$ and having then necessarily the limit v^+. Indeed,

$$\int_\Omega |v_n^+ - v^+|^p \, dx \leq \int_\Omega |v_n - v|^p \, dx = \int_\Omega |u_n - u|^p \, dx \, , \tag{1.10}$$

where we used the fact that the constant of Lipschitz continuity of F is equal to 1. Relation (1.10) is nothing other than the strong continuity of $u \mapsto v^+$ in $L^p(\Omega)$. Further we have,

$$\begin{aligned}
&\int_\Omega |\nabla v_n^+ - \nabla v^+|^p \, dx \\
&\qquad = \int_\Omega |H(v_n)\nabla v_n - H(v)\nabla v|^p \, dx \\
&\qquad \leq c \int_\Omega |H(v_n)|^p |\nabla v_n - \nabla v|^p + |H(v_n) - H(v)|^p |\nabla v|^p \, dx \\
&\qquad \leq c \int_\Omega |\nabla u_n - \nabla u|^p \, dx + c \int_\Omega |H(v_n) - H(v)|^p |\nabla u|^p \, dx \\
&\qquad = A_n + B_n \, ,
\end{aligned} \tag{1.11}$$

where we used $|H(\xi)| \leq 1$ and $\nabla v_n = \nabla u_n$, $\nabla v = \nabla u$, respectively. The sequence A_n converges to zero due to $u_n \to u$ in $W^{1,p}(\Omega)$, while B_n converges to zero due to the Lebesgue dominated convergence theorem. Therefore we have $F(v_n) \to F(v)$ in $W^{1,p}(\Omega)$ which completes the proof. ∎

Remark 1.12 Note that for the special case of $a_0 = 0$ we can extend the previous lemma also for $G(\xi) \equiv |\xi|$. Thus we have for $v \in W^{1,p}(\Omega)$,

$$\frac{\partial}{\partial x_i}|v| = \operatorname{sgn}(v) \frac{\partial v}{\partial x_i} \qquad \text{a.e. in } \Omega \, . \tag{1.13}$$

Lemma 1.14 *Let $T > 0$ and let $a_0 \geq 0$ be a given constant. Let $u \in W(T) \equiv \big\{u \in L^2(0,T;W^{1,2}(\mathbb{R}^d)), \frac{\partial u}{\partial t} \in L^2(0,T;W^{-1,2}(\mathbb{R}^d))\big\}$. Set $v \equiv u - a_0$ again. Then*

$$v^+ \in L^2(0,T;W^{1,2}(\mathbb{R}^d)) \cap C(0,T;L^2(\mathbb{R}^d))\,, \tag{1.15}$$

and, for almost all $0 \leq t_1 < t_2 \leq T$,

$$2\int_{t_1}^{t_2} \Big\langle \frac{\partial v}{\partial t}(s), v^+(s)\Big\rangle\, ds = \|v^+(t_2)\|^2_{L^2(\mathbb{R}^d)} - \|v^+(t_1)\|^2_{L^2(\mathbb{R}^d)}. \tag{1.16}$$

Proof : The assumption $u \in W(T)$ implies for almost all $t \in (0,T)$ that $u(t) \in W^{1,2}(\mathbb{R}^d)$ and thus Lemma 1.1 gives $v^+(t) \in W^{1,2}(\mathbb{R}^d)$. Further we have for $F(\xi) \equiv \max(\xi, 0)$ that

$$|F(\xi - a_0)| \leq |\xi|\,, \quad |F'(\xi)| \leq 1\,. \tag{1.17}$$

Formulae (1.8) and (1.17) imply

$$\|v^+(t)\|_{W^{1,2}(\mathbb{R}^d)} \leq \|u(t)\|_{W^{1,2}(\mathbb{R}^d)} \qquad \text{a.e. in } (0,T)\,. \tag{1.18}$$

Thus we obtain $v^+ \in L^2(0,T;W^{1,2}(\mathbb{R}^d))$ and

$$\|v^+\|_{L^2(0,T;W^{1,2}(\mathbb{R}^d))} \leq \|u\|_{L^2(0,T;W^{1,2}(\mathbb{R}^d))}.$$

Let us first show (1.16) for $u \in W^{1,2}((0,T) \times \mathbb{R}^d)$: Lemma 1.1 implies $v^+ \in W^{1,2}((0,T) \times \mathbb{R}^d)$ and since $v^+ F'(v) = v^+$ (almost everywhere), we obtain

$$v^+ \frac{\partial v^+}{\partial t} = v^+ F'(v) \frac{\partial v}{\partial t} = v^+ \frac{\partial v}{\partial t} \quad \text{a.e. in } (0,T) \times \mathbb{R}^d\,. \tag{1.19}$$

From (1.19) it follows that

$$\Big\langle \frac{\partial v}{\partial t}, v^+ \Big\rangle = \int_{\mathbb{R}^d} \frac{\partial v}{\partial t}\, v^+\, dx = \int_{\mathbb{R}^d} \frac{\partial v^+}{\partial t}\, v^+\, dx\,,$$

which, together with the partial integration formula (2.46) from Chapter 1 for $u = v = v^+$, implies (1.16) for $u \in W^{1,2}((0,T) \times \mathbb{R}^d)$. Now we use the density of $W^{1,2}((0,T) \times \mathbb{R}^d)$ in $W(T)$. Let $u \in W(T)$ and let $\{u^n\} \subseteq W^{1,2}((0,T) \times \mathbb{R}^d)$ converge strongly in $W(T)$ to u. Let $v^n \equiv u^n - a_0$. We want to show that

$$(v^n)^+ \to v^+ \text{ in } L^2(0,T;W^{1,2}(\Omega)) \cap C(0,T;L^2(\mathbb{R}^d)). \tag{1.20}$$

We know that $W(T) \hookrightarrow C(0,T;L^2(\mathbb{R}^d))$ and therefore it follows (see (1.17)) that

$$\|(v^n)^+(t)\|_{L^2(\mathbb{R}^d)} \leq \|u^n(t)\|_{L^2(\mathbb{R}^d)} \qquad \forall\, t \in (0,T)\,, \tag{1.21}$$

$$\|(v^n)^+(t) - (v^n)^+(s)\|_{L^2(\mathbb{R}^d)} \leq \|u^n(t) - u^n(s)\|_{L^2(\mathbb{R}^d)}. \tag{1.22}$$

From (1.21) and (1.22) we find that $(v^n)^+ \in C(0,T;L^2(\mathbb{R}^d))$. In the same way we obtain

$$\max_{[0,T]} \|(v^n)^+(t) - v^+(t)\|_{L^2(\mathbb{R}^d)} \leq \max_{[0,T]} \|u^n(t) - u(t)\|_{L^2(\mathbb{R}^d)}$$

which implies

$$(v^n)^+ \to v^+ \qquad \text{in} \qquad C(0,T;L^2(\mathbb{R}^d)). \tag{1.23}$$

However, Lemma 1.7 shows that $u(t) \mapsto v^+(t)$ is a continuous mapping in the space $W^{1,2}(\mathbb{R}^d)$, which implies

$$\|(v^n)^+(t) - v^+(t)\|_{W^{1,2}(\mathbb{R}^d)} \to 0 \qquad \text{for a.a. } t \in [0,T]\,.$$

From this and (1.18) we obtain

$$(v^n)^+ \to v^+ \qquad \text{in } L^2(0,T;W^{1,2}(\mathbb{R}^d))\,. \tag{1.24}$$

In order to finish the proof we must justify the limiting process in

$$2\int_{t_1}^{t_2} \Big\langle \frac{\partial v^n}{\partial t}(s), (v^n)^+(s) \Big\rangle\, ds = \|(v^n)^+(t_2)\|^2_{L^2(\mathbb{R}^d)} - \|(v^n)^+(t_1)\|^2_{L^2(\mathbb{R}^d)}.$$

This is possible in the first term due to the boundedness of $\frac{\partial v^n}{\partial t}$ ($= \frac{\partial u^n}{\partial t}$) in the space $L^2(0,T;W^{-1,2}(\mathbb{R}^d))$ and due to (1.24). On the right-hand side we use (1.23). ∎

A.2 Parabolic theory

Let us consider the following Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \varepsilon\Delta u + \lambda u &= f \qquad \text{in } (0,T)\times\mathbb{R}^d\,, \\ u(0,\cdot) &= u^0 \qquad \text{in } \mathbb{R}^d\,, \end{aligned} \tag{2.1}$$

where $T \in (0,\infty]$, $\lambda \geq 0$, $\varepsilon > 0$. We have the following existence and regularity result.

Theorem 2.2 *Let $u^0 \in L^2(\mathbb{R}^d)$, $f \in L^2\big(0,T;W^{-1,2}(\mathbb{R}^d)\big)$. Then (2.1) has a unique solution $u \in W(T) \equiv \big\{u \in L^2(0,T;W^{1,2}(\mathbb{R}^d));\ \frac{\partial u}{\partial t} \in L^2(0,T;W^{-1,2}(\mathbb{R}^d))\big\}$, satisfying (2.1) in the following sense: for almost all $t \in (0,T)$ we have*

$$\Big\langle \frac{\partial u}{\partial t}(t), \varphi \Big\rangle + \varepsilon \int_{\mathbb{R}^d} \nabla u(t)\, \nabla \varphi \, dx + \int_{\mathbb{R}^d} \lambda u \varphi \, dx = \langle f(t), \varphi \rangle$$

for all $\varphi \in W^{1,2}(\mathbb{R}^d)$.

PROOF : See LIONS AND MAGENES [1972a, Chapter 3]. ■

Remark 2.3 Recall that $W(T) \hookrightarrow C(0,T;L^2(\mathbb{R}^d))$ and therefore the initial condition $(2.1)_2$ is meaningful in the classical sense.

Theorem 2.4 *Assume that for some $m \geq 0$, $m \in \mathbb{N}$, we have $u^0 \in W^{m,2}(\mathbb{R}^d)$, $f \in L^2(0,T;W^{m-1,2}(\mathbb{R}^d))$. Then the solution of (2.1) satisfies*

$$\begin{aligned} & u \in L^2(0,T;W^{m+1,2}(\mathbb{R}^d)) \cap C(0,T;W^{m,2}(\mathbb{R}^d))\,, \\ & \frac{\partial u}{\partial t} \in L^2(0,T;W^{m-1,2}(\mathbb{R}^d))\,. \end{aligned} \tag{2.5}$$

PROOF : See LIONS AND MAGENES [1972b, Chapter 4]. ■

In the case of a smooth bounded domain $\Omega \subset \mathbb{R}^d$ and $T \in (0,\infty)$ we consider

$$\begin{aligned} \frac{\partial u}{\partial t} - \varepsilon \Delta u + \lambda u = f \qquad & \text{in } (0,T) \times \Omega\,, \\ u(0,\cdot) = u^0 \qquad & \text{in } \Omega\,, \\ u = u^D \qquad & \text{on } (0,T) \times \partial\Omega\,, \end{aligned} \tag{2.6}$$

where u^0 and u^D are supposed to satisfy appropriate compatibility conditions. Then we have the following theorem.

Theorem 2.7 *Let $u^0 \in L^2(\Omega)$, $u^D \in L^2(0,T;W^{1/2,2}(\partial\Omega)) \cap W^{1/4,2}(0,T;L^2(\partial\Omega))$, $f \in L^2(0,T;W^{-1,2}(\Omega))$. Then there exists a unique solution $u \in W(T) \equiv \big\{u \in L^2(0,T;W^{1,2}(\Omega));\ \frac{\partial u}{\partial t} \in L^2(0,T;W^{-1,2}(\Omega))\big\}$, satisfying (2.6) in the following sense: for almost all $t \in (0,T)$ we have*

$$\Big\langle \frac{\partial u}{\partial t}(t), \varphi \Big\rangle + \varepsilon \int_{\Omega} \nabla u(t)\, \nabla \varphi \, dx + \int_{\Omega} \lambda u \varphi \, dx = \langle f(t), \varphi \rangle$$

for all $\varphi \in W_0^{1,2}(\Omega)$*. Moreover, for almost every* $t \in (0,T)$*,*†

$$u(t) - \widetilde{u}^D(t) \in W_0^{1,2}(\Omega)\,. \tag{2.8}$$

PROOF : See LIONS AND MAGENES [1972b, Chapter 4]. ■

Remark 2.9 For the initial condition $(2.6)_2$ we argue in the same way as in Remark 2.3.

Theorem 2.10 *Let* $f \in L^2(0,T;W^{m-1,2}(\Omega))$*,* $u^0 \in W^{m,2}(\Omega)$ *and* $u^D \in L^2(0,T;W^{m+1/2,2}(\partial\Omega)) \cap W^{m+1/4,2}(0,T;L^2(\partial\Omega))$ *for some* $m \in \mathbb{N} \cup \{0\}$*. Then the solution of (2.6) satisfies*

$$\begin{aligned} &u \in L^2(0,T;W^{m+1,2}(\Omega)) \cap C(0,T;W^{m,2}(\Omega))\,, \\ &\frac{\partial u}{\partial t} \in L^2(0,T;W^{m-1,2}(\Omega))\,. \end{aligned} \tag{2.11}$$

PROOF : See LIONS AND MAGENES [1972b, Chapter 4]. ■

A.3 Ordinary differential equations

Let us consider for $\mathbf{c} : I_\delta \equiv (t_0 - \delta, t_0 + \delta) \longrightarrow \mathbb{R}^N$, the system of ordinary differential equations

$$\begin{aligned} \frac{d}{dt}\mathbf{c}(t) &= \mathbf{F}(t, \mathbf{c}(t))\,, \qquad t \in I_\delta\,, \\ \mathbf{c}(t_0) &= \mathbf{c}_0 \in \mathbb{R}^N\,. \end{aligned} \tag{3.1}$$

Assume $\mathbf{F} : I_\delta \times K \longrightarrow \mathbb{R}^N$, where $K \equiv \{\mathbf{c} \in \mathbb{R}^N, |\mathbf{c} - \mathbf{c}_0| < \Delta\}$ for some $\Delta > 0$.

Definition 3.2 *A function* $\mathbf{F} : I_\delta \times K \longrightarrow \mathbb{R}^N$ *is said to satisfy the* **Carathéodory conditions** *if*

- $t \mapsto F_i(t, \mathbf{c})$ *is measurable for all* $i = 1, \dots, N$ *and for all* $\mathbf{c} \in K$*,*
- $\mathbf{c} \mapsto F_i(t, \mathbf{c})$ *is continuous for almost all* $t \in I_\delta$*,*
- *there exists an integrable function* $G : I_\delta \longrightarrow \mathbb{R}$ *such that*

$$|F_i(t, \mathbf{c})| \le G(t) \quad \forall\, (t, \mathbf{c}) \in I_\delta \times K\,, \quad \forall\, i = 1, \dots, N\,. \tag{3.3}$$

Theorem 3.4 *Let* $\mathbf{F}$ *satisfy the Carathéodory conditions. Then there exist* $\delta' \in (0, \delta)$ *and a continuous function* $\mathbf{c} : I_{\delta'} \longrightarrow \mathbb{R}^N$ *such that*

† From the assumption on u^D it follows that there is a $\widetilde{u}^D(t) \in W^{1,2}(\Omega)$ such that $u^D(t) = \widetilde{u}^D(t)$ on $\partial\Omega$ for almost all $t \in (0,T)$ in the sense of traces. This observation makes (2.8) meaningful.

- $\frac{d\mathbf{c}}{dt}$ *exists for almost all* $t \in I_{\delta'}$,
- $\mathbf{c}$ *solves (3.1).*

PROOF : See for example CODDINGTON AND LEVINSON [1955, Chapter 2] or WALTER [1970, Chapter 1]. ■

Lemma 3.5 (Gronwall) *Let* $y : (0,T) \to \mathbb{R}$ *and* $g : (0,T) \to \mathbb{R}$ *be non-negative functions,* $g \in L^1(0,T)$. *Let the inequality*

$$y(t) \le C + \int_0^t g(s)\, y(s)\, ds \tag{3.6}$$

hold for $t \in (0,T)$ *with* $C \in \mathbb{R}$. *Then*

$$y(t) \le C \exp \int_0^t g(s)\, ds\,, \quad t \in (0,T)\,. \tag{3.7}$$

PROOF : See for example WALTER [1970, Chapter 1]. ■

A.4 Bases consisting of eigenfunctions of an elliptic operator

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. We will investigate (construct) bases of spaces of divergence-free functions. For $s \ge 1$ and $p > 1$ let us define

$$\mathcal{V} \equiv \{\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^d; \operatorname{div} \boldsymbol{\varphi} = 0\}; \tag{4.1}$$

$$H \equiv \text{the closure of } \mathcal{V} \text{ in the } L^2(\Omega)^d\text{-norm}\,; \tag{4.2}$$

$$V_p \equiv \text{the closure of } \mathcal{V} \text{ in } W^{1,p}(\Omega)^d \text{ with } \|\nabla(\cdot)\|_p\text{-norm}\,; \tag{4.3}$$

$$V^s \equiv \text{the closure of } \mathcal{V} \text{ in } W^{s,2}(\Omega)^d\text{-norm}\,. \tag{4.4}$$

If $s = 1$ or $p = 2$, then V will denote the spaces V_2, V^1, respectively. The scalar product in H is marked by $(\cdot,\cdot)$ while the scalar product in V^s is marked by $(\!(\cdot,\cdot)\!)_s$.

Remark 4.5 The spaces V_p and V^s can be characterized as follows:

$$V_p = \{\mathbf{u} \in W_0^{1,p}(\Omega)^d;\ \operatorname{div} \mathbf{u} = 0\}\,,$$
$$V^s = \{\mathbf{u} \in W^{s,2}(\Omega)^d;\ \gamma(\mathbf{u}) = \mathbf{0} \text{ at } \partial\Omega,\ \operatorname{div} \mathbf{u} = 0\}\,,$$

where γ is a trace operator defined in (2.14)–(2.15) in Chapter 1.

The characterization of H requires a definition of an appropriate trace operator. Let

$$\gamma : W^{1,2}(\Omega) \hookrightarrow H^{1/2}(\partial\Omega)$$

denote an usual trace operator introduced in (2.14)–(2.15) in Chapter 1. By $H^{-1/2}(\partial\Omega)^d$ we mean the dual space of $H^{1/2}(\partial\Omega)^d$, i.e., the space $\left(H^{1/2}(\partial\Omega)^d\right)^*$. Defining

$$E(\Omega) \equiv \left\{\mathbf{u} \in L^2(\Omega)^d; \operatorname{div}\mathbf{u} \in L^2(\Omega)\right\},$$

it is possible to construct a trace operator

$$\widehat{\gamma} : E(\Omega) \hookrightarrow H^{-1/2}(\partial\Omega)^d,$$

such that $\widehat{\gamma}(\mathbf{u}) = \mathbf{u}\cdot\mathbf{n}$ for $\mathbf{u} \in C^1(\overline{\Omega})$, $\mathbf{n}$ being an outer normal vector. Then it holds that

$$H = \left\{\mathbf{u} \in L^2(\Omega)^d, \widehat{\gamma}(\mathbf{u}) = \mathbf{0}, \operatorname{div}\mathbf{u} = 0 \text{ in } \mathcal{D}'(\Omega)\right\}. \tag{4.6}$$

We refer to Constantin and Foias [1988], Temam [1977] and Galdi [1994a, 1994b] for proofs of these and other detailed results about the space H.

In some parts of the book, we consider spaces of space-periodic functions. For this purpose, it will be necessary to change slightly the definition of $\mathcal{V}$ in (4.1) to

$$\mathcal{V} = \{\boldsymbol{\varphi} \in C^\infty_{\mathrm{per}}(\Omega)^d; \operatorname{div}\boldsymbol{\varphi} = 0, \int_\Omega \boldsymbol{\varphi}\,dx = \mathbf{0}\}, \tag{4.7}$$

where Ω is a cube in $\mathbb{R}^d$.

Let us introduce the projector

$$\mathbb{P} : L^2(\Omega)^d \longrightarrow H,$$

sometimes called **Leray's operator**.

Definition 4.8 *The operator $A \equiv -\mathbb{P}\Delta\mathbf{u} : H \longrightarrow H$ with the domain of definition $D(A) = W^{2,2}(\Omega)^d \cap V$ is called the* **Stokes operator**.

It is worth recalling that in general (for Dirichlet boundary conditions) the operators $\mathbb{P}$ and Δ do not commute. But if the space periodic problem is taken into consideration, then

$$A\mathbf{u} = -\mathbb{P}\Delta\mathbf{u} = -\Delta\mathbf{u}. \tag{4.9}$$

We will study the following spectral problem: find $\boldsymbol{\omega}^r \in V^s$ and $\lambda_r \in \mathbb{R}$ satisfying

$$((\boldsymbol{\omega}^r, \boldsymbol{\varphi}))_s = \lambda_r(\boldsymbol{\omega}^r, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in V^s. \tag{4.10}$$

Theorem 4.11 *There exists a countable set $\{\lambda_r\}_{r=1}^{\infty}$ and a corresponding family of eigenvectors $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ solving the problem (4.10) such that*

- $(\boldsymbol{\omega}^r, \boldsymbol{\omega}^s) = \delta_{rs} \qquad \forall\, r, s \in \mathbb{N}$,
- $1 \le \lambda_1 \le \lambda_2 \le \dots$ *and* $\lambda_r \to \infty$ *as* $r \to \infty$,
- $((\frac{\boldsymbol{\omega}^r}{\sqrt{\lambda_r}}, \frac{\boldsymbol{\omega}^s}{\sqrt{\lambda_s}}))_s = \delta_{rs} \qquad \forall\, r, s \in \mathbb{N}$,
- $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ *forms a basis in* V^s.

Moreover, defining $H^N \equiv \operatorname{span}\{\boldsymbol{\omega}^1, \dots, \boldsymbol{\omega}^N\}$ *(a linear hull) and* $P^N(\mathbf{v}) \equiv \sum_{i=1}^N (\mathbf{v}, \boldsymbol{\omega}^r)\, \boldsymbol{\omega}^r : V^s \longrightarrow H^N$, *we obtain*

$$\|P^N\|_{\mathcal{L}(V^s, V^s)} \le 1, \qquad \|P^N\|_{\mathcal{L}((V^s)^*, (V^s)^*)} \le 1, \tag{4.12}$$

$$\|P^N\|_{\mathcal{L}(H,H)} \le 1. \tag{4.13}$$

Remark 4.14 If we consider instead of (4.10) the problem of finding $\boldsymbol{\omega}^r \in W_0^{s,2}(\Omega)^d$ and $\lambda_r \in \mathbb{R}$ satisfying

$$((\boldsymbol{\omega}^r, \boldsymbol{\varphi}))_s = \lambda_r(\boldsymbol{\omega}^r, \boldsymbol{\varphi}), \quad \forall\, \boldsymbol{\varphi} \in W_0^{s,2}(\Omega)^d, \tag{4.15}$$

all conditions of Theorem 4.11 hold, too. A particular case, $s = 1$, is used in Chapter 4 of this book.

Proof (of Theorem 4.11):

- *The existence of* $\boldsymbol{\omega}^1$

  Define

$$\frac{1}{\lambda_1} \equiv \sup_{\|\mathbf{v}\|_{s,2} \le 1} (\mathbf{v}, \mathbf{v}). \tag{4.16}$$

(Note that $\frac{1}{\lambda_1} \le 1$). Then there exists a sequence $\{\mathbf{v}^k\}_{k=1}^{\infty}$ such that

$$(\mathbf{v}^k, \mathbf{v}^k) \to \frac{1}{\lambda_1} \quad \text{and} \quad \|\mathbf{v}^k\|_{s,2} = 1.$$

Consequently, there exists $\boldsymbol{\omega}^1 \in V^s$ and a subsequence $\{\mathbf{v}^{k'}\} \subset \{\mathbf{v}^k\}$ such that

$$\mathbf{v}^{k'} \rightharpoonup \boldsymbol{\omega}^1 \quad \text{in } V^s \quad \text{and } \mathbf{v}^{k'} \to \boldsymbol{\omega}^1 \quad \text{in } H.$$

Clearly, $\|\boldsymbol{\omega}^1\|_{s,2} \le 1$. In fact, $\|\boldsymbol{\omega}^1\|_{s,2} = 1$. Indeed, if $\|\boldsymbol{\omega}^1\|_{s,2} < 1$ then $\boldsymbol{\omega} \equiv \boldsymbol{\omega}^1/\|\boldsymbol{\omega}^1\|_{s,2}$ fulfills

$$\|\boldsymbol{\omega}\|_{s,2} = 1 \quad \text{and} \quad (\boldsymbol{\omega}, \boldsymbol{\omega}) = \frac{(\boldsymbol{\omega}^1, \boldsymbol{\omega}^1)}{\|\boldsymbol{\omega}^1\|_{s,2}} > \frac{1}{\lambda_1},$$

which contradicts the definition of λ_1, see (4.16). It remains to show that $\boldsymbol{\omega}^1$ is an eigenvector.

Let $\mathbf{h} \in V^s$. Defining

$$\Phi(t) \equiv \frac{(\boldsymbol{\omega}^1 + t\mathbf{h}, \boldsymbol{\omega}^1 + t\mathbf{h})}{((\boldsymbol{\omega}^1 + t\mathbf{h}, \boldsymbol{\omega}^1 + t\mathbf{h}))_s},$$

we obtain

$$\begin{aligned} 0 = \frac{d}{dt}\,\Phi(t)\big|_{t=0} &= \frac{2\,(\boldsymbol{\omega}^1, \mathbf{h})\,((\boldsymbol{\omega}^1, \boldsymbol{\omega}^1))_s - 2\,(\boldsymbol{\omega}^1, \boldsymbol{\omega}^1)\,((\boldsymbol{\omega}^1, \mathbf{h}))_s}{((\boldsymbol{\omega}^1, \boldsymbol{\omega}^1))_s^2} \\ &= \frac{2\,(\boldsymbol{\omega}^1, \mathbf{h}) - \frac{2}{\lambda_1}\,((\boldsymbol{\omega}^1, \mathbf{h}))_s}{((\boldsymbol{\omega}^1, \boldsymbol{\omega}^1))_s^2}. \end{aligned}$$

This implies

$$\lambda_1\,(\boldsymbol{\omega}^1, \mathbf{h}) = ((\boldsymbol{\omega}^1, \mathbf{h}))_s \qquad \forall\, h \in V^s.$$

- *Iterative construction*

Let us assume that the existence of the first N, $N \ge 1$, eigenvectors $\{\boldsymbol{\omega}^i\}_{i=1}^N$ and corresponding eigenvalues $\{\lambda_i\}_{i=1}^N$ has already been proved. Define

$$W^N \equiv \{\mathbf{v} \in V^s; ((\mathbf{v}, \boldsymbol{\omega}^i))_s = 0,\ i = 1, \dots, N\}.$$

Then the same construction as in the first part of the proof allows to find $\boldsymbol{\omega}^{N+1}$ such that

$$(\boldsymbol{\omega}^{N+1}, \boldsymbol{\omega}^{N+1}) = \sup_{\substack{\|\mathbf{v}\|_{s,2}=1 \\ \mathbf{v} \in W^N}} (\mathbf{v}, \mathbf{v}) \equiv \frac{1}{\lambda_{N+1}}. \tag{4.17}$$

From this construction we see that

$$1 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{N+1} \le \dots, \tag{4.18}$$

$$(\boldsymbol{\omega}^r, \boldsymbol{\omega}^s) = 0 \quad \text{if } r \ne s, \tag{4.19}$$

$$((\boldsymbol{\omega}^r, \boldsymbol{\omega}^s))_s = \delta_{rs} \qquad \forall\, r, s \in \mathbb{N}. \tag{4.20}$$

• $\lambda_r \to \infty$ *as* $r \to \infty$

Let $\lim_{r\to\infty} \lambda_r = \lambda < \infty$. Since $\|\boldsymbol{\omega}^r\|_{s,2} = 1$, we have

$$\boldsymbol{\omega}^{r_k} \to \boldsymbol{\omega} \quad \text{in } H\,, \qquad \text{as } k \to \infty\,. \tag{4.21}$$

Simple calculations give

$$\begin{aligned} 2 = (\!(\boldsymbol{\omega}^{r_k} - \boldsymbol{\omega}^{r_\ell}, \boldsymbol{\omega}^{r_k} - \boldsymbol{\omega}^{r_\ell})\!)_s = \lambda_{r_k}&(\boldsymbol{\omega}^{r_k}, \boldsymbol{\omega}^{r_k} - \boldsymbol{\omega}^{r_\ell}) \\ &- \lambda_{r_\ell}(\boldsymbol{\omega}^{r_\ell}, \boldsymbol{\omega}^{r_k} - \boldsymbol{\omega}^{r_\ell})\,, \end{aligned}$$

and due to boundedness of λ_{r_k} and (4.21), we can make the right-hand side arbitrarily small letting $k, \ell \to \infty$.

• $\{\lambda_r\}_{r=1}^{\infty}$ *are all eigenvalues of (4.10)*

Let us assume for contradiction the existence of $\lambda \in \mathbb{R}$ and $\boldsymbol{\omega} \in V^s$ such that $\lambda \neq \lambda_r$ for all $r \in \mathbb{N}$, $\|\boldsymbol{\omega}\|_{s,2} = 1$ and $(\!(\boldsymbol{\omega}, \boldsymbol{\varphi})\!)_s = \lambda(\boldsymbol{\omega}, \boldsymbol{\varphi})$ for all $\boldsymbol{\varphi} \in V^s$. Clearly, there is an $i \in \mathbb{N}$ such that $\lambda_i < \lambda < \lambda_{i+1}$. Since $(\!(\boldsymbol{\omega}^k, \boldsymbol{\omega})\!)_s = \lambda_k(\boldsymbol{\omega}^k, \boldsymbol{\omega})$ and $(\!(\boldsymbol{\omega}, \boldsymbol{\omega}^k)\!) = \lambda(\boldsymbol{\omega}, \boldsymbol{\omega}^k)$ for all $k = 1, \ldots, i$, we see that $(\boldsymbol{\omega}, \boldsymbol{\omega}^k) = 0$ for all $k = 1, \ldots, i$. Therefore $\boldsymbol{\omega} \in W^i$. However,

$$(\boldsymbol{\omega}, \boldsymbol{\omega}) = \frac{1}{\lambda} > \frac{1}{\lambda_{i+1}} = \sup_{\substack{\|\mathbf{v}\|_{s,2}=1 \\ \mathbf{v}\in W^i}} (\mathbf{v}, \mathbf{v})\,,$$

which is a contradiction.

• $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ *is a basis in* V^s

Put $X \equiv \operatorname{span}\{\boldsymbol{\omega}^1, \ldots, \boldsymbol{\omega}^N, \ldots\}$. Assume that $X \neq V^s$. Therefore there exists $\boldsymbol{\Psi} \in V^s$, $\|\boldsymbol{\Psi}\|_{s,2} = 1$ such that $(\!(\boldsymbol{\Psi}, \boldsymbol{\omega}^r)\!)_s = 0$ for all $r \in \mathbb{N}$. Then we have for all $r \in \mathbb{N}$,

$$(\boldsymbol{\Psi}, \boldsymbol{\Psi}) \le \sup_{\substack{\mathbf{v}\in W^r \\ \|\mathbf{v}\|_{s,2}=1}} (\mathbf{v}, \mathbf{v}) = \frac{1}{\lambda_r}\,.$$

Since $\frac{1}{\lambda_r} \to 0$, $\boldsymbol{\Psi}$ has to be zero element of V^s. This contradicts the fact that $\|\boldsymbol{\Psi}\|_{s,2} = 1$.

• *Renormalization of basis*

We have already proved that $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ satisfies (4.19), (4.20) and (4.10). Setting $\widehat{\boldsymbol{\omega}}^r \equiv \boldsymbol{\omega}^r/\sqrt{\lambda_r}$ we see that $\widehat{\boldsymbol{\omega}}^r$ solves again (4.10) with eigenvalues λ_r. Moreover, $\widehat{\boldsymbol{\omega}}^r$ are orthonormal in H.

Replacing $\{\boldsymbol{\omega}^r\}_{r=1}^{\infty}$ by $\{\widehat{\boldsymbol{\omega}}^r\}_{r=1}^{\infty}$ the proof of the first four assertions is finished.

• *The continuity of P^N*

Consider now $\mathbf{v} \in V^s$. Then

$$\|P^N(\mathbf{v})\|_{s,2}^2 = \sum_{i=1}^{N} (\mathbf{v}, \boldsymbol{\omega}^i)^2 ((\boldsymbol{\omega}^i, \boldsymbol{\omega}^i))_s = \sum_{i=1}^{N} \frac{((\mathbf{v}, \boldsymbol{\omega}^i))_s^2}{\lambda_i^2} ((\boldsymbol{\omega}^i, \boldsymbol{\omega}^i))_s$$
$$\leq \sum_{i=1}^{N} \left(\left(\mathbf{v}, \frac{\boldsymbol{\omega}^i}{\sqrt{\lambda_i}}\right)\right)_s^2 \leq \|\mathbf{v}\|_{s,2}^2 .$$

Thus (4.12) is proved, since P^N is a selfadjoint operator. Finally, (4.13) follows immediately.

The proof of Theorem 4.11 is complete. ■

We will consider henceforth the eigenvalue problem (4.10) for the special case $s = 1$, i.e. $V^1 = V$. Then the eigenvalue problem (4.10) is equivalent to the problem of finding eigenvalues (and eigenvectors) for the Stokes operator A defined in (4.8), if $\partial\Omega$ is smooth (C^2, for example), and we have

$$\lambda_r(\boldsymbol{\omega}^r, \mathbf{u}^N) = (A\boldsymbol{\omega}^r, \mathbf{u}^N) = (\nabla\boldsymbol{\omega}^r, \nabla\mathbf{u}^N) . \tag{4.22}$$

Let us consider the so-called Stokes system, studied in detail in Galdi [1994a, 1994b]:

$$\begin{aligned} -\Delta\mathbf{u} + \nabla p = \mathbf{F} &\quad \text{in } \Omega , \\ \operatorname{div} \mathbf{u} = 0 &\quad \text{in } \Omega , \\ \mathbf{u} = 0 &\quad \text{at } \partial\Omega . \end{aligned} \tag{4.23}$$

If $\partial\Omega \in C^2$, then from the regularity result for the Stokes system follows (let us refer again to Galdi [1994a, 1994b] and references therein) that for all $\mathbf{F} \in L^2(\Omega)^d$,

$$\|\mathbf{u}\|_{2,2} + \|p\|_{1,2} \leq c \, \|\mathbf{F}\|_2 \ , \tag{4.24}$$

where $c = c(\Omega, d)$. Taking $\mathbf{F} = A\mathbf{u} \in H$, we have in particular (see also Proposition 4.7 in Constantin and Foias [1988]),

$$\|\mathbf{u}\|_{2,2} \leq c\|A\mathbf{u}\|_2 \qquad \forall\, \mathbf{u} \in D(A) . \tag{4.25}$$

From Theorem 4.11 it follows that the operators P^N are continuous in $V = V^1$. The inequality (4.25) allows us to prove the continuity of P^N also in $D(A)$.

Lemma 4.26 *Let $A = -\mathbb{P}\Delta\mathbf{u}$, $D(A)$ and P^N be as above. Then there exists $C = C(\Omega, d) > 0$ such that*

$$\|P^N\mathbf{u}\|_{2,2} \leq C \, \|\mathbf{u}\|_{2,2} \quad \forall\, \mathbf{u} \in D(A) . \tag{4.27}$$

PROOF : Let us first prove that $AP^N(\mathbf{u}) = P^N A\mathbf{u}$ a.e. in Ω. It suffices to show that

$$(AP^N \mathbf{u}, \mathbf{v}) = (P^N A\mathbf{u}, \mathbf{v}) \qquad \forall\, \mathbf{v} \in D(A)\,. \tag{4.28}$$

However, $(AP^N \mathbf{u}, \mathbf{v}) = ((P^N \mathbf{u}, \mathbf{v}))_1$ and

$$\begin{aligned}
(P^N A\mathbf{u}, \mathbf{v}) &= \sum_{r=1}^{N} (A\mathbf{u}, \boldsymbol{\omega}^r)\,(\boldsymbol{\omega}^r, \mathbf{v}) = \sum_{r=1}^{N} \frac{1}{\lambda_r}(A\mathbf{u}, \boldsymbol{\omega}^r)\,((\boldsymbol{\omega}^r, \mathbf{v}))_1 \\
&= \sum_{r=1}^{N} \frac{1}{\lambda_r}((\mathbf{u}, \boldsymbol{\omega}^r))_1\,((\boldsymbol{\omega}^r, \mathbf{v}))_1 = \sum_{r=1}^{N} (\boldsymbol{\omega}^r, \mathbf{u})\,((\boldsymbol{\omega}^r, \mathbf{v}))_1 \\
&= ((P^N \mathbf{u}, \mathbf{v}))_1\,.
\end{aligned}$$

So (4.28) holds.

Now we have

$$\|P^N \mathbf{u}\|_{2,2} \overset{(4.25)}{\leq} c\,\|AP^N \mathbf{u}\|_2 = c\,\|P^N A\mathbf{u}\|_2 \overset{(4.13)}{\leq} c\,\|A\mathbf{u}\|_2\,.$$

But $\|A\mathbf{u}\|_2 \leq \|\mathbf{u}\|_{2,2}$, and (4.27) follows. ∎