CHAPTER 4

Measure-valued solutions and nonlinear hyperbolic equations

4.1 Introduction

Let $Q \subset \mathbb{R}^d$ be a measurable set, on which a sequence of measurable functions $\mathbf{z}^j : Q \longrightarrow \mathbb{R}^s$, $j = 1, 2, \ldots$, is defined. Let $\tau : \mathbb{R}^s \longrightarrow \mathbb{R}$ be a continuous function. We are interested in describing the behaviour of $\tau(\mathbf{z}^j)$ as $j \to \infty$. As already pointed out in Theorem 3.2.3, the behaviour of $\tau(\mathbf{z}^j)$ can be represented by a measure-valued function $\nu : Q \longrightarrow \mathrm{Prob}(\mathbb{R}^s)$ provided that the sequence $\{\mathbf{z}^j\}_{j=1}^{\infty}$ is uniformly bounded in $L^\infty(Q)^s$. More precisely, since $\tau(\mathbf{z}^j)$ is also uniformly bounded in $L^\infty(Q)$, there exists a subsequence still denoted by $\mathbf{z}^j$ and a function $\overline{\tau} \in L^\infty(Q)$ such that

$$\tau(\mathbf{z}^j) \overset{*}{\rightharpoonup} \overline{\tau} \qquad \text{weakly-}^* \text{ in } L^\infty(Q)\,, \tag{1.1}$$

and, according to Theorem 2.3 in Chapter 3, we have

$$\overline{\tau}(y) = \langle \nu_y, \tau \rangle = \int_{\mathbb{R}^s} \tau(\boldsymbol{\lambda})\, d\nu_y(\boldsymbol{\lambda})\,, \tag{1.2}$$

for almost all $y \in Q$. Here we use the notation $\nu_y \equiv \nu(y)$. Moreover, the measures ν_y are uniformly compactly supported. Note that since the values of $\mathbf{z}^j$ belong to some compact set of $\mathbb{R}^s$, we could suppose, without loss of generality, that $\tau \in C_C(\mathbb{R}^s)$.

In this chapter we extend the characterization (1.2) to a broader class of functions τ and $\mathbf{z}^j$. Following Ball [1989], we prove a more general version of Theorem 2.3 in Chapter 3. As a consequence of this generalization, we obtain the following result: If Q is a bounded set and $\mathbf{z}^j$ are uniformly bounded in $L^p(Q)^s$ for some $p \in (1, \infty)$, then there exists a subsequence still denoted by $\mathbf{z}^j$ and a function $\nu : Q \longrightarrow \mathrm{Prob}(\mathbb{R}^s)$ such that for all $\tau \in C(\mathbb{R}^s)$,

$$\begin{aligned} \tau(\mathbf{z}^j) &\rightharpoonup \overline{\tau} \quad \text{weakly in } L^{\frac{p}{p-1}}(Q)\,, \\ \overline{\tau}(y) &= \langle \nu_y, \tau \rangle \quad \text{for a.a. } y \in Q\,, \end{aligned} \tag{1.3}$$

whenever τ satisfies the growth condition

$$\left|\tau(\boldsymbol{\xi})\right| \leq c\left(1+|\boldsymbol{\xi}|\right)^{p-1}, \qquad \boldsymbol{\xi} \in \mathbb{R}^s . \tag{1.4}$$

As we will see, mainly in the next chapter, the nonlinearities satisfying (1.4) occur naturally in problems of continuum mechanics. However, we will use the above-mentioned theorem later in this chapter (see Section 4.3). Using again the method of vanishing viscosity, we prove the global in time existence of a measure-valued solution to the scalar hyperbolic equation of second order:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x_j} a_j(\nabla u) = f \qquad \text{in } I \times \Omega\,, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded open set and $I = (0,T)$. The equation (1.5) is considered together with Dirichlet boundary conditions and initial conditions for u and $\frac{\partial u}{\partial t}$. From the physical point of view, the unknown function $u : Q_T \to \mathbb{R}$ can be interpreted as a displacement of a vibrating membrane. The functions $a_i : \mathbb{R}^d \longrightarrow \mathbb{R}$ are considered to have linear growth, i.e., $p = 2$ in (1.4). More precisely, we assume that a_i are represented by a scalar quadratic potential.

Recall that equation (1.5) can be rewritten as a hyperbolic system of $(d+1)$ equations. Indeed, putting $v_0 \equiv \frac{\partial u}{\partial t}$ and $v_j \equiv \frac{\partial u}{\partial x_j}$, we obtain for $\widehat{\mathbf{v}} \equiv (v_1, \ldots, v_d)$, $\mathbf{v} \equiv (v_0, \widehat{\mathbf{v}})$,

$$\begin{aligned} &\frac{\partial v_0}{\partial t} - \frac{\partial}{\partial x_j} a_j(\widehat{\mathbf{v}}) = f\,, \\ &\qquad \frac{\partial v_i}{\partial t} - \frac{\partial v_0}{\partial x_i} = 0\,, \quad i = 1, \ldots, d\,. \end{aligned} \tag{1.6}$$

See Section 1.1.3 in Chapter 1 for details.

Finally, let us emphasize that, to the knowledge of the authors, (global in time) existence of a weak solution to (1.5) is still an open problem. This is one of the reasons why the concept of a measure-valued solution to problem (1.5) is introduced. Within this class we obtain an existence result (see Theorem 3.34) and thus the question of the existence of a weak solution to (1.5) can be reformulated as a question of whether the measures ν_y are Dirac ones (compare with Chapter 3).

4.2 A version of the fundamental theorem on Young measures

Following BALL [1989], we prove in this section a version of the theorem about Young measures. This theorem will play a key role in the construction of measure-valued solutions not only in the next section, but also in the following chapter. Let us recall that the symbol $L^\infty_\omega(Q; M(\mathbb{R}^s))$ represents the space of all weakly measurable functions $\nu : Q \longrightarrow M(\mathbb{R}^s)$, for which the norm $\operatorname{ess\,sup}_{y\in Q} \|\nu_y\|_{M(\mathbb{R}^s)}$ is finite (see Chapter 1 and Definition 2.7 in Chapter 3 for more details).

Theorem 2.1 *Let $Q \subset \mathbb{R}^d$ be a measurable set and let $\mathbf{z}^j : Q \to \mathbb{R}^s$, $j = 1, 2, \ldots$, be a sequence of measurable functions. Then there exists a subsequence still denoted by $\mathbf{z}^j$ and a measure-valued function ν with the following properties:*

1. *The function ν satisfies*

$$\nu \in L^\infty_\omega(Q; M(\mathbb{R}^s)), \tag{2.2}$$

$$\|\nu_y\|_{M(\mathbb{R}^s)} \le 1 \qquad \text{for a.e. } y \in Q\,, \tag{2.3}$$

and we have for every $\varphi \in C_0(\mathbb{R}^s)$, as $j \to \infty$,

$$\varphi(\mathbf{z}^j) \overset{*}{\rightharpoonup} \overline{\varphi} \quad \text{weakly-}^* \text{ in } L^\infty(Q)\,, \qquad \overline{\varphi}(y) \overset{a.e.}{=} \langle \nu_y, \varphi\rangle\,. \tag{2.4}$$

2. *Moreover, if*

$$\lim_{k\to\infty} \sup_{j=1,2,\ldots} \operatorname{meas}\left\{y \in Q \cap B_R\,;\, |\mathbf{z}^j(y)| \ge k\right\} = 0 \tag{2.5}$$

for every $R > 0$, where $B_R \equiv \{y \in Q\,;\, |y| \le R\}$, then

$$\|\nu_y\|_{M(\mathbb{R}^s)} = 1 \qquad \text{for a.e. } y \in Q. \tag{2.6}$$

3. *Let $\Psi : [0, \infty) \longrightarrow \mathbb{R}$ be a Young function satisfying the Δ_2-condition.*† *If condition (2.5) holds and if we have for some continuous function $\tau : \mathbb{R}^s \longrightarrow \mathbb{R}$*

$$\sup_{j=1,2,\ldots} \int_Q \Psi\big(|\tau(\mathbf{z}^j)|\big)\, dy < \infty\,, \tag{2.7}$$

† Definitions and properties of Young functions as well as Orlicz spaces can be found in Chapter 1.

then

$$\tau(\mathbf{z}^j) \overset{*}{\rightharpoonup} \overline{\tau} \quad \text{weakly-}^* \text{ in } L_\Psi(Q)\,, \qquad \overline{\tau}(y) \overset{a.e.}{=} \langle \nu_y, \tau\rangle\,. \tag{2.8}$$

Before proving Theorem 2.1 let us make the following remarks.

Remark 2.9 If $\mathbf{z}^j$ are uniformly bounded in $L^p(Q)^s$ for some $p \in [1,\infty)$, the condition (2.5) is satisfied. Indeed, denoting $A_k^j \equiv \{y \in Q \cap B_R; |\mathbf{z}^j(y)| \geq k\}$, we have

$$|A_k^j|\, k^p \leq \int_{A_k^j} |\mathbf{z}^j(y)|^p\, dy \leq \int_Q |\mathbf{z}^j(y)|^p\, dy \leq c\,.$$

Since c is independent of both j and k we obtain

$$\sup_{j=1,2,\dots} |A_k^j| \leq \frac{c}{k^p}\,,$$

which implies (2.5).

Corollary 2.10 *Let $Q \subset \mathbb{R}^d$ be a bounded open set. Let $\mathbf{z}^j$ be uniformly bounded in $L^p(Q)^s$. Then there exists a subsequence still denoted by $\mathbf{z}^j$ and a measure-valued function ν, such that for all $\tau : \mathbb{R}^s \longrightarrow \mathbb{R}$ satisfying for some $q > 0$ the growth condition*

$$\left|\tau(\boldsymbol{\xi})\right| \leq c\left(1 + |\boldsymbol{\xi}|\right)^q \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d\,, \tag{2.11}$$

we have

$$\begin{aligned} \tau(\mathbf{z}^j) \rightharpoonup \overline{\tau} \quad &\text{weakly in } L^r(Q)\,,\\ \overline{\tau}(y) &\overset{a.e.}{=} \langle \nu_y, \tau\rangle\,, \end{aligned} \tag{2.12}$$

provided that

$$1 < r \leq \frac{p}{q}\,. \tag{2.13}$$

PROOF : Due to Remark 2.9 the condition (2.5) of Theorem 2.1 is satisfied. Let us verify (2.7). We have (for the Young function $\Psi(u) = u^r$)

$$\int_Q \Psi\left(|\tau(\mathbf{z}^j)|\right) dy = \int_Q |\tau(\mathbf{z}^j)|^r\, dy \overset{(2.11)}{\leq} c^r \int_Q \left(1 + |\mathbf{z}^j|\right)^{qr} dy$$

and the last integral is uniformly bounded (with respect to j) if $qr \leq p$. The lower bound $r > 1$ follows from the properties of Orlicz functions, namely from $\lim_{s\to\infty} \frac{\Psi(s)}{s} = \infty$. ■

We immediately see from (2.13) that we can easily find an $r \in (1, \frac{p}{q}]$ if $p > q$. The case $p = q$ is not covered by the above result. The investigation of this situation can be found in KINDERLEHRER AND PEDREGAL [1992b].

PROOF (of Theorem 2.1):

Ad 1: With $\mathbf{z}^j$ we associate the mapping $\nu^j : Q \longrightarrow M(\mathbb{R}^s)$ defined for almost all $y \in Q$ by

$$\nu_y^j = \delta_{\mathbf{z}^j(y)}\,, \tag{2.14}$$

where $\delta_{\mathbf{z}}$ is a Dirac measure supported at $\mathbf{z} \in \mathbb{R}^s$. Hence,

$$\|\nu_y^j\|_{M(\mathbb{R}^s)} = 1 \qquad \text{for a.e. } y \in Q. \tag{2.15}$$

Since $C_0(\mathbb{R}^s)$ and consequently also $L^1(Q; C_0(\mathbb{R}^s))$ are separable, we obtain from Theorem 2.11 in Chapter 3 the existence of a subsequence still denoted by ν^j and an element $\nu \in L^\infty_\omega(Q; M(\mathbb{R}^s))$ such that

$$\nu^j \overset{*}{\rightharpoonup} \nu \qquad \text{weakly-}^* \text{ in } L^\infty_\omega(Q; M(\mathbb{R}^s)). \tag{2.16}$$

Therefore, taking in (2.16) test functions $h \in L^1(Q; C_0(\mathbb{R}^s))$ in the form of $h(y, \boldsymbol{\lambda}) = g(y)\varphi(\boldsymbol{\lambda})$ with $g \in L^1(Q)$ and $\varphi \in C_0(\mathbb{R}^s)$, we see that

$$\langle \nu_y^j, \varphi\rangle \overset{*}{\rightharpoonup} \langle \nu_y, \varphi\rangle \qquad \text{weakly-}^* \text{ in } L^\infty(Q) \tag{2.17}$$

for every $\varphi \in C_0(\mathbb{R}^s)$. Hence, (2.4) is proved, and (2.3) follows from the weak-* lower semicontinuity of the norm $\|\cdot\|_{M(\mathbb{R}^s)}$ and (2.15).

Ad 2: We prove that ν is almost everywhere a probability measure, provided that (2.5) holds. Let us define, for $\boldsymbol{\lambda} \in \mathbb{R}^s$,

$$\vartheta^k(\boldsymbol{\lambda}) \equiv \begin{cases} 1 & |\boldsymbol{\lambda}| \le k\,, \\ 1 + k - |\boldsymbol{\lambda}| & k \le |\boldsymbol{\lambda}| \le k+1\,, \\ 0 & |\boldsymbol{\lambda}| \ge k+1\,. \end{cases} \tag{2.18}$$

Let $E \subset Q$ be an arbitrary bounded measurable set and let $|E|$ denote its Lebesgue measure in $\mathbb{R}^d$. Then, for fixed $k \in \mathbb{N}$, we denote

$$A_k^j \equiv \left\{ y \in E; |\mathbf{z}^j(y)| \ge k \right\}. \tag{2.19}$$

Then

$$0 \le \frac{1}{|E|}\int_E \left(1 - \vartheta^k(\mathbf{z}^j(y))\right)\, dy \\ \le \frac{|A_k^j|}{|E|} \le \sup_{j=1,2,\ldots} \frac{|A_k^j|}{|E|} \equiv \varepsilon_k\,, \tag{2.20}$$

where $\varepsilon_k \to 0$ if $k \to \infty$ due to (2.5). Therefore

$$1 - \varepsilon_k \le \frac{1}{|E|}\int_E \vartheta^k(\mathbf{z}^j(y))\, dy\,. \tag{2.21}$$

Then, letting $j \to \infty$ in (2.21) and using (2.17), we obtain

$$1 - \varepsilon_k \le \frac{1}{|E|}\int_E \langle \nu_y, \vartheta^k\rangle\, dy \le \frac{1}{|E|}\int_E \|\nu_y\|_{M(\mathbb{R}^s)}\, dy \overset{(2.3)}{\le} 1. \tag{2.22}$$

Thus,

$$1 = \frac{1}{|E|}\int_E \|\nu_y\|_{M(\mathbb{R}^s)}\, dy, \tag{2.23}$$

which implies that $\|\nu_y\|_{M(\mathbb{R}^s)} = 1$ for almost all $y \in Q$ and (2.6) is proved.

Ad 3: It remains to prove (2.8). Let $\tau : \mathbb{R}^s \to \mathbb{R}$ be a continuous function satisfying (2.7). We can suppose without loss of generality that $\tau \ge 0$. Taking ϑ^k as in (2.18), we put

$$\tau^k(\mathbf{z}) \equiv \tau(\mathbf{z})\vartheta^k(\mathbf{z})\,. \tag{2.24}$$

Choose Φ such that Φ and Ψ are complementary Young functions. According to Lemma 2.30 in Chapter 1 we have $(C_\Phi(\Omega))^* = L_\Psi(\Omega)$. In order to prove (2.8) one must show that

$$\langle \nu_y, \tau\rangle \in L_\Psi(Q) \tag{2.25}$$

and verify the following limiting processes for an arbitrary $g \in C_\Phi(Q)$:

$$\int_Q g(y)\,\tau^k(\mathbf{z}^j(y))\, dy \overset{k\to\infty}{\longrightarrow} \int_Q g(y)\,\tau(\mathbf{z}^j(y))\, dy\,, \tag{2.26}$$

$$\int_Q g(y)\tau^k(\mathbf{z}^j(y))\, dy \overset{j\to\infty}{\longrightarrow} \int_Q g(y)\,\langle \nu_y, \tau^k\rangle\, dy\,, \tag{2.27}$$

$$\int_Q g(y)\langle \nu_y, \tau^k\rangle\, dy \overset{k\to\infty}{\longrightarrow} \int_Q g(y)\langle \nu_y, \tau\rangle\, dy\,, \tag{2.28}$$

where the convergence in (2.26) is meant to be uniform with respect to j. Thus, using the triangle inequality, one can see

$$\int_Q g(y)\,\tau(\mathbf{z}^j(y))\,dy \overset{j\to\infty}{\longrightarrow} \int_Q g(y)\langle \nu_y, \tau\rangle\,dy \qquad \text{for all} \quad g \in C_\Phi(Q)\,,$$

which is exactly (2.8). Let us show (2.25)–(2.28). The fact that the convergence in (2.26) is uniform (with respect to j) can be proved directly. Indeed, for A_k^j as in (2.19),

$$\begin{aligned}
&\int_Q |g(y)|\;|\tau^k(\mathbf{z}^j(y)) - \tau(\mathbf{z}^j(y))|\,dy \\
&\qquad \le \int_{A_k^j} |g(y)|\,(1-\vartheta^k(\mathbf{z}^j(y)))\tau(\mathbf{z}^j(y))\,dy \\
&\qquad \overset{(2.29),\ \text{Ch.1}}{\le} \|g\|_{L_\Phi(A_k^j)}\|\tau(\mathbf{z}^j)\|_{L_\Psi(A_k^j)} \\
&\qquad \overset{(2.28),\ \text{Ch.1}}{\le} \|g\|_{L_\Phi(A_k^j)}\left(1+\int_Q \Psi\big(|\tau(\mathbf{z}^j)|\big)\,dy\right) \\
&\qquad \overset{(2.7)}{\le} C\,\|g\|_{L_\Phi(A_k^j)}\,.
\end{aligned} \tag{2.29}$$

By virtue of (2.5) and the absolute continuity of the Orlicz norm $\|g\|_{L_\Phi(Q')}$ with respect to Q' (see Theorem 2.40 in Chapter 1), one finds that for every $\varepsilon > 0$ there exists a k_0 such that for $k > k_0$ the right-hand side of (2.29) is less than ε. Thus (2.26) is proved.

The limiting process in (2.27) follows immediately from (2.4) and the imbedding $C_\Phi(Q) \hookrightarrow L_1(Q)$.

To verify (2.28), note that $\tau^{k+1} \ge \tau^k \ge 0$. Due to the monotone convergence theorem we have for almost all $y \in Q$, as $k \to \infty$,

$$\langle \nu_y, \tau^k\rangle = \int_{\mathbb{R}^s} \tau^k d\nu_y \longrightarrow \int_{\mathbb{R}^s} \tau\,d\nu_y = \langle \nu_y, \tau\rangle\,. \tag{2.30}$$

For $g \ge 0$ we have

$$g(y)\,\langle \nu_y, \tau^{k+1}\rangle \;\ge\; g(y)\,\langle \nu_y, \tau^k\rangle \;\ge\; 0 \tag{2.31}$$

and from (2.30)

$$\lim_{k\to\infty} g(y)\langle \nu_y, \tau^k\rangle = g(y)\langle \nu_y, \tau\rangle \qquad \text{for a.e. } y \in Q. \tag{2.32}$$

Again, due to the monotone convergence theorem, we obtain (2.28).

Note that $\int_Q g(y)\langle \nu_y, \tau\rangle\,dy$ can still be infinite. Its boundedness is a consequence of (2.25), which will be proved in what follows.

Since $\tau^k \le \tau^{k+1} \le \tau$, we have for fixed k,

$$\sup_{j=1,2,\dots} \int_Q \Psi(|\tau^k(\mathbf{z}^j)|)\,dy \le \sup_{j=1,2,\dots} \int_Q \Psi(|\tau(\mathbf{z}^j)|)\,dy \le c\,, \quad (2.33)$$

or

$$\|\tau^k(\mathbf{z}^j)\|_{L_\Psi(Q)} = \|\langle \nu^j, \tau^k\rangle\|_{L_\Psi(Q)} \le c\,, \quad (2.34)$$

where c depends neither on k nor on j. Thus there exist $a_k \in L_\Psi(Q)$ such that, for $j \to \infty$,

$$\langle \nu^j, \tau^k\rangle \overset{*}{\rightharpoonup} a_k \qquad \text{weakly-}^* \text{ in } L_\Psi(Q). \quad (2.35)$$

The comparison of (2.35) with (2.27) gives $a_k(y) = \langle \nu_y, \tau^k\rangle$ for almost every $y \in Q$. Moreover, $a_k = \langle \nu, \tau^k\rangle$ belongs to the ball in $L_\Psi(Q)$ given by (2.34), independently of k. Consequently,

$$\|\langle \nu, \tau^k\rangle\|_{L_\Psi(Q)} \le \text{ c}. \quad (2.36)$$

Then there exists $a \in L_\Psi(Q)$ such that

$$\int_Q g(y)\,\langle \nu_y, \tau^k\rangle\,dy \longrightarrow \int_Q g(y)a(y)\,dy \quad (2.37)$$

for every $g \in C_\Phi(Q)$, as $k \to \infty$. Comparing (2.37) with (2.28) for smooth test functions g, we obtain

$$a(y) = \langle \nu_y, \tau\rangle \qquad \text{for a.e. } y \in Q \quad (2.38)$$

and (2.25) is proved. The proof of Theorem 2.1 is complete. ∎

Remark 2.39 In Ball [1989] the author claims that the theorem can be regarded as a consequence of a general lower semicontinuity theorem proved by Balder [1984].

The original statement of Theorem 2.1 in Ball [1989] is a bit different. Instead of (2.7) it is assumed that $\{\tau(\mathbf{z}^j)\}_{j=1}^\infty$ is sequentially weakly relatively compact in $L^1(Q)$. In fact, this condition is equivalent to (2.7) if Q is bounded and $\Psi : [0,\infty) \longrightarrow \mathbb{R}$ is a continuous function such that $\lim_{|\xi|\to\infty} \frac{\Psi(\xi)}{\xi} = \infty$. In our case, Ψ is moreover a Young function, i.e., a convex one. If we omit this assumption, we only have

$$\tau(\mathbf{z}^j) \rightharpoonup \langle \nu, \tau\rangle \qquad \text{weakly in } L^1(Q)\,,$$

which corresponds to the assertion of the theorem in Ball [1989].

4.3 Measure-valued solutions to a hyperbolic equation of second order

The subject of this section is to prove the existence of measure-valued solutions (global in time) to the following problem.

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with a smooth boundary $\partial\Omega$. Let $T \in (0,\infty)$, $I \equiv (0,T)$ and $Q_T \equiv I \times \Omega$. Given functions $f : Q_T \longrightarrow \mathbb{R}$, u_0, $u_1 : \Omega \longrightarrow \mathbb{R}$, we look for a function $u : Q_T \longrightarrow \mathbb{R}$, solving

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x_j} a_j(\nabla u) = f \qquad \text{in } Q_T\,, \tag{3.1}$$

$$u(0,\cdot) = u_0\,, \quad \frac{\partial u}{\partial t}(0,\cdot) = u_1 \qquad \text{in } \Omega\,, \tag{3.2}$$

$$u = 0 \qquad \text{on } I \times \partial\Omega\,, \tag{3.3}$$

provided that there exists a function $\vartheta \in C^2(\mathbb{R}^d)$, called **potential** to $\boldsymbol{a} = (a_1, \ldots, a_d)$, and positive constants α, $\beta \in \mathbb{R}$ such that for all $i, j = 1, \ldots, d$,

$$\frac{\partial \vartheta}{\partial \xi_i} = a_i\,, \tag{3.4}$$

$$\vartheta(\mathbf{0}) = \frac{\partial \vartheta}{\partial \xi_i}(\mathbf{0}) = 0\,, \tag{3.5}$$

$$\left| \frac{\partial^2 \vartheta}{\partial \xi_i \partial \xi_j} \right| \leq \beta\,, \tag{3.6}$$

and

$$\alpha |\boldsymbol{\eta}|^2 \leq \frac{\partial^2 \vartheta(\boldsymbol{\xi})}{\partial \xi_k \partial \xi_\ell} \eta_k \eta_\ell \tag{3.7}$$

for all $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathbb{R}^d$.

A simple calculation (see also Section 5.1) shows that (3.4)–(3.7) imply

$$|a_i(\boldsymbol{\xi})| \leq \beta |\boldsymbol{\xi}|\,, \tag{3.8}$$

$$\frac{\alpha}{2} |\boldsymbol{\xi}|^2 \leq \vartheta(\boldsymbol{\xi}) \leq \frac{\beta}{2} |\boldsymbol{\xi}|^2\,, \tag{3.9}$$

$$|a_i(\boldsymbol{\xi}) - a_i(\boldsymbol{\eta})| \leq \beta |\boldsymbol{\xi} - \boldsymbol{\eta}|\,, \tag{3.10}$$

for all $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathbb{R}^d$ and $i = 1, \ldots, d$.

Let us suppose, for a moment, that u^ε are some approximations of the problem (3.1)–(3.7) having the following properties:

$$\nabla u^\varepsilon \ \text{ is uniformly bounded in } \ L^2(Q_T)^d \,, \tag{3.11}$$

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} \rightharpoonup \frac{\partial^2 u}{\partial t^2} \ \text{ weakly in } \ L^2(I; W^{-1,2}(\Omega)) \,. \tag{3.12}$$

Due to (3.11) we can use Theorem 2.1 or rather its Corollary 2.10 with $p = 2$, $q = 1$ and $r = 2$ (for $s = d$) and we obtain the existence of a measure-valued function $\nu : Q_T \longrightarrow \mathrm{Prob}(\mathbb{R}^d)$ such that for each $i = 1, \dots, d$,

$$a_i(\nabla u^\varepsilon) \rightharpoonup \overline{a_i} \qquad \text{weakly in } L^2(Q_T)$$

with

$$\overline{a_i}(t,x) \stackrel{a.e.}{=} \langle \nu_{t,x}, a_i \rangle \ ,$$

at least for a subsequence. We see that under assumptions (3.11), (3.12) there is a hope that we will be able to find a limiting process for solutions u^ε of a suitable approximate problem to a measure-valued solution—a couple (u, ν)—of the original problem (3.1)–(3.7).

Now we are in a position to define the notion of a measure-valued solution to the problem (3.1)–(3.7).

Definition 3.13 *Let*

$$f \in L^2(Q_T) \,, \quad u_0 \in W_0^{1,2}(\Omega) \,, \quad u_1 \in L^2(\Omega) \tag{3.14}$$

be given. A couple (u, ν) such that

$$\begin{aligned} u &\in L^2(I; W_0^{1,2}(\Omega)) \,, \\ \frac{\partial u}{\partial t} &\in L^2(Q_T) \,, \\ \frac{\partial^2 u}{\partial t^2} &\in L^2(I; W^{-1,2}(\Omega)) \,, \\ \nu &\in L^\infty_\omega\big(Q_T; \mathrm{Prob}\,(\mathbb{R}^d)\big) \,, \end{aligned} \tag{3.15}$$

is called a **measure-valued solution** *to the problem (3.1)–(3.7) if and only if*

$$u(0) = u_0 \,, \qquad \frac{\partial u}{\partial t}(0) = u_1 \,, \tag{3.16}$$

and the following identities are fulfilled:

$$\frac{\partial u}{\partial x_j}(t,x) = \int_{\mathbb{R}^d} \sigma_j \, d\nu_{t,x}(\boldsymbol{\sigma}) \tag{3.17}$$

for almost all $(t,x) \in Q_T$, *and*

$$\int_0^T \Big(\frac{\partial^2 u}{\partial t^2}(t), \varphi(t)\Big)\, dt + \int_{Q_T} \frac{\partial \varphi}{\partial x_i}(t,x)\, \langle \nu_{t,x}, a_i \rangle \, dx\, dt = \int_0^T \big(f(t), \varphi(t)\big)\, dt \tag{3.18}$$

for all $\varphi \in L^2\big(I; W_0^{1,2}(\Omega)\big)$.

Remark 3.19

- We have $u \in C(I; L^2(\Omega))$ and $\frac{\partial u}{\partial t} \in C(I; W^{-1,2}(\Omega))$ due to Lemma 2.45 in Chapter 1. Hence, the initial conditions (3.16) are understood in the sense of the spaces $L^2(\Omega)$, $W^{-1,2}(\Omega)$, respectively, for example

$$\lim_{t \to 0+} \|u(t) - u_0\|_2 = 0\,.$$

- In this section we will denote by $(\cdot,\cdot)$ not only the usual L^2 scalar product, but also the duality between the spaces $W_0^{1,2}(\Omega)$ and $W^{-1,2}(\Omega)$. The reason is that $\langle \cdot, \cdot \rangle$ denotes here the duality between $M(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$ (see (3.18)).

In order to prove the existence of a measure-valued solution, we need to find convenient approximations, denoted by u^ε, having (at least) the properties (3.11) and (3.12). For this purpose we perturb the equation (3.1) by the term $-\varepsilon \Delta \frac{\partial u}{\partial t}$ and we will investigate the following problem.

Find, for each $\varepsilon > 0$, a function $u^\varepsilon : Q_T \longrightarrow \mathbb{R}$ satisfying

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - \frac{\partial}{\partial x_j} a_j(\nabla u^\varepsilon) - \varepsilon \Delta \frac{\partial u^\varepsilon}{\partial t} = f \quad \text{in } Q_T\,, \tag{3.20}$$

$$u^\varepsilon(0,\cdot) = u_0\,, \quad \frac{\partial u^\varepsilon}{\partial t}(0,\cdot) = u_1 \quad \text{in } \Omega\,, \tag{3.21}$$

$$u^\varepsilon = 0 \quad \text{on } I \times \partial\Omega\,. \tag{3.22}$$

Definition 3.23 *Let* f, u_0, u_1 *fulfill (3.14). A function* u^ε, *defined on* Q_T, *is called a weak solution to the problem (3.20)–(3.22),* *(3.4)–(3.7), if and only if*

$$u^\varepsilon \in L^2\big(I; W_0^{1,2}(\Omega)\big)\,, \tag{3.24}$$

$$\frac{\partial u^\varepsilon}{\partial t} \in L^2\big(I; W_0^{1,2}(\Omega)\big)\,, \tag{3.25}$$

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} \in L^2\big(I; W^{-1,2}(\Omega)\big)\,, \tag{3.26}$$

and the following equality

$$\begin{aligned} &\int_0^T \Big(\frac{\partial^2 u^\varepsilon}{\partial t^2}(\tau), \varphi(\tau)\Big)\, d\tau + \int_0^T \Big(a_i(\nabla u^\varepsilon(\tau)), \frac{\partial \varphi(\tau)}{\partial x_i}\Big)\, d\tau \\ &+ \varepsilon \int_0^T \Big(\frac{\partial}{\partial t}\frac{\partial u^\varepsilon}{\partial x_i}(\tau), \frac{\partial \varphi}{\partial x_i}(\tau)\Big)\, d\tau = \int_0^T \big(f(\tau), \varphi(\tau)\big)\, d\tau \end{aligned} \tag{3.27}$$

is fulfilled for all $\varphi \in L^2\big(I; W_0^{1,2}(\Omega)\big)$.

Remark 3.28 The initial conditions (3.21) with assumptions (3.14) are meaningful because of Lemma 2.45 in Chapter 1. In fact, $u^\varepsilon \in C\big(I; W_0^{1,2}(\Omega)\big)$ and $\frac{\partial u^\varepsilon}{\partial t} \in C\big(I; L^2(\Omega)\big)$.

Theorem 3.29 *Let f, u_0 and u_1 satisfy (3.14). Then for every $\varepsilon > 0$ there exists just one weak solution u^ε to the problem (3.20)–(3.22), (3.4)–(3.7). Moreover, u^ε satisfies the following uniform estimates:*

$$\|u^\varepsilon\|_{L^\infty(I; W_0^{1,2}(\Omega))} \le c\,, \tag{3.30}$$

$$\Big\|\frac{\partial u^\varepsilon}{\partial t}\Big\|_{L^\infty(I; L^2(\Omega))} \le c\,, \tag{3.31}$$

$$\varepsilon \int_0^T \Big\|\nabla \frac{\partial u^\varepsilon}{\partial t}(\tau)\Big\|_2^2\, d\tau \le c\,, \tag{3.32}$$

$$\Big\|\frac{\partial^2 u^\varepsilon}{\partial t^2}\Big\|_{L^2(I; W^{-1,2}(\Omega))} \le c\,. \tag{3.33}$$

Although we use the standard Galerkin method to solve the parabolic problem (3.20)–(3.22), we present a detailed proof here, since it shows another method of obtaining the almost everywhere convergence of ∇u^ε, needed for the limiting process in the nonlinearities a_i, $i = 1, \dots, d$. However, in order to concentrate on our main goal, i.e., to prove the existence of a measure-valued solution to our hyperbolic problem, we postpone the proof of Theorem 3.29 to the end of this section.

Theorem 3.34 *Let f, u_0, u_1 satisfy (3.14). Then there exists a measure-valued solution to the problem (3.1)–(3.7).*

PROOF : Due to the uniform estimates (3.30)–(3.33), we can choose a subsequence $\{u^{\varepsilon_k}\} \subset \{u^\varepsilon\}$ such that

$$u^{\varepsilon_k} \rightharpoonup u \qquad \text{weakly in } L^2(I; W_0^{1,2}(\Omega)) \,, \tag{3.35}$$

$$\frac{\partial u^{\varepsilon_k}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \qquad \text{weakly in } L^2(Q_T) \,, \tag{3.36}$$

$$\frac{\partial^2 u^{\varepsilon_k}}{\partial t^2} \rightharpoonup \frac{\partial^2 u}{\partial t^2} \qquad \text{weakly in } L^2(I; W^{-1,2}(\Omega)) \,, \tag{3.37}$$

as $k \to \infty$.

Furthermore, u^{ε_k} are weak solutions to the approximate problem (3.20)–(3.22), (3.4)–(3.7), i.e., they satisfy the equality (3.27) for all $\varphi \in L^2(0,T; W_0^{1,2}(\Omega))$. Let us investigate the limiting process in (3.27).

Due to (3.37),

$$\int_0^T \Big(\frac{\partial^2 u^{\varepsilon_k}}{\partial t^2}(\tau), \varphi(\tau)\Big)\, d\tau \to \int_0^T \Big(\frac{\partial^2 u}{\partial t^2}(\tau), \varphi(\tau)\Big)\, d\tau \tag{3.38}$$

and according to (3.32),

$$\sqrt{\varepsilon_k} \int_0^T \sqrt{\varepsilon_k}\Big(\nabla \frac{\partial u^{\varepsilon_k}}{\partial t}, \nabla\varphi\Big)\, d\tau \to 0 \,, \tag{3.39}$$

as $\varepsilon_k \to 0+$. Considering the limit of $\int_0^T \big(a_i(\nabla u^{\varepsilon_k}(t), \frac{\partial \varphi(t)}{\partial x_i}\big)\, dx\, dt$, we apply Corollary 2.10. Since $\|\nabla u^{\varepsilon_k}\|_{L^2(Q_T)} \le c$ due to (3.30), we take $\mathbf{z}^j = \nabla u^{\varepsilon_j}$, $p = 2$, $q = 1$ and $r = 2$, $s = d$ in the above cited corollary. Thus we get the existence of a measure-valued function $\nu \in L^\infty_\omega(Q_T; \mathrm{Prob}(\mathbb{R}^d))$ such that for $\tau = a_i$,

$$a_i(\nabla u^{\varepsilon_k}) \rightharpoonup \overline{a_i} \qquad \text{weakly in } L^2(Q_T),$$

where

$$\overline{a_i}(t,x) \stackrel{\text{a.e.}}{=} \int_{\mathbb{R}^d} a_i(\boldsymbol{\sigma})\, d\nu_{t,x}(\boldsymbol{\sigma}) \,, \quad i = 1, \ldots, d.$$

Therefore,

$$\int_{Q_T} a_i(\nabla u^{\varepsilon_k}) \frac{\partial \varphi}{\partial x_i}\, dx\, dt \to \int_{Q_T} \frac{\partial \varphi}{\partial x_i} \langle \nu_{t,x}, a_i\rangle\, dx\, dt \,. \tag{3.40}$$

Using (3.38)–(3.40) in (3.27), we can conclude that the couple (u, ν)

satisfies (3.18) for all $\varphi \in L^2(0,T;W_0^{1,2}(\Omega))$. It remains to verify (3.17). Due to (3.35),

$$\int_{Q_T} \nabla u^{\varepsilon_k} \varphi \, dx \, dt \overset{k\to\infty}{\longrightarrow} \int_{Q_T} \nabla u \, \varphi \, dx \, dt \tag{3.41}$$

for all $\varphi \in \mathcal{D}(Q_T)$. Using again Corollary 2.10 for τ equal to components of $\mathbf{Id}$, (the other parameters are the same as above), we obtain

$$\int_{Q_T} \nabla u^{\varepsilon_k} \varphi \, dx \, dt \to \int_{Q_T} \varphi \, \langle \nu_{t,x}, \mathbf{Id} \rangle \, dx \, dt \tag{3.42}$$

for all $\varphi \in \mathcal{D}(Q_T)$. Hence,

$$\int_{Q_T} \varphi(t,x) \langle \nu_{t,x}, \mathbf{Id} \rangle \, dx \, dt = \int_{Q_T} \varphi(t,x) \nabla u(t,x) \, dx \, dt$$

for all $\varphi \in \mathcal{D}(Q_T)$, which implies (3.17). ∎

In the rest of this section we will prove Theorem 3.29. For simplicity we drop the superscript ε.

Proof (of Theorem 3.29):

Uniqueness: Let v, w be two weak solutions of the problem (3.20)–(3.22), (3.4)–(3.7). Putting $u \equiv v - w$, we have for all $t \in (0,T)$ the identity

$$\begin{aligned} &\int_0^t \Big(\frac{\partial^2 u}{\partial t^2}(\tau), \varphi(\tau) \Big) \, d\tau \\ &\qquad + \int_0^t \Big(a_i(\nabla v(\tau)) - a_i(\nabla w(\tau)), \frac{\partial \varphi(\tau)}{\partial x_i} \Big) \, d\tau \\ &\qquad + \varepsilon \int_0^t \Big(\frac{\partial}{\partial x_i} \frac{\partial u}{\partial t}(\tau), \frac{\partial \varphi(\tau)}{\partial x_i} \Big) \, d\tau = 0 \end{aligned} \tag{3.43}$$

for all $\varphi \in L^2\big(0,t;W_0^{1,2}(\Omega)\big)$. Taking $\frac{\partial u}{\partial t}$ as a test function in (3.43), we obtain

$$\begin{aligned} &\frac{1}{2} \Big\| \frac{\partial u}{\partial t}(t) \Big\|_2^2 - \frac{1}{2} \Big\| \frac{\partial u}{\partial t}(0) \Big\|_2^2 + \varepsilon \int_0^t \Big\| \nabla \frac{\partial u}{\partial t}(\tau) \Big\|_2^2 \, d\tau \\ &\quad = \int_0^t \Big(a_i(\nabla w(\tau)) - a_i(\nabla v(\tau)), \frac{\partial}{\partial x_i} \frac{\partial u}{\partial t}(\tau) \Big) \, d\tau \equiv I_1 \, , \end{aligned}$$

or (since $\frac{\partial u}{\partial t}(0) = 0$),

$$\varepsilon \int_0^t \Big\| \nabla \frac{\partial u}{\partial t}(\tau) \Big\|_2^2 \, d\tau \le |I_1| \, . \tag{3.44}$$

Because of (3.10), the right-hand side of (3.44) is estimated by Hölder's and Young's inequalities as follows:

$$|I_1| \le \frac{\varepsilon}{2}\int_0^t \big\|\nabla\frac{\partial u}{\partial t}(\tau)\big\|_2^2\,d\tau + \frac{\beta^2}{2\varepsilon}\int_0^t \|\nabla u(\tau)\|_2^2\,d\tau\,, \tag{3.45}$$

which gives

$$\int_0^t \big\|\nabla\frac{\partial u}{\partial t}(\tau)\big\|_2^2\,d\tau \le \frac{\beta^2}{\varepsilon^2}\int_0^t \|\nabla u(\tau)\|_2^2\,d\tau\,. \tag{3.46}$$

Relation $u(\tau)=\int_0^\tau \frac{\partial u}{\partial t}(s)ds$ allows us to rewrite (3.46) in the form

$$\int_0^t \big\|\nabla\frac{\partial u}{\partial t}(\tau)\big\|_2^2\,d\tau \le \frac{\beta^2 T}{\varepsilon^2}\int_0^t \Big(\int_0^\tau \big\|\nabla\frac{\partial u}{\partial t}(s)\big\|_2^2\,ds\Big)\,d\tau\,. \tag{3.47}$$

If we define $y(t)\equiv\int_0^t \|\nabla\frac{\partial u}{\partial t}(\tau)\|_2^2\,d\tau$ then (3.47) reads

$$y(t) \le \frac{\beta^2 T}{\varepsilon^2}\int_0^t y(\tau)\,d\tau\,,$$

and Gronwall's lemma 3.5 in the Appendix finishes the proof. In fact, we obtain

$$\int_0^t \big\|\nabla\frac{\partial u}{\partial t}(\tau)\big\|_2^2\,d\tau = 0 \quad \forall\, t\in(0,T]\,.$$

If we again use the relation $u(t)=\int_0^t \frac{\partial u}{\partial t}(s)\,ds$, we obtain

$$\|\nabla u(t)\|_2 = 0 \qquad \forall\, t\in(0,T]\,, \tag{3.48}$$

which implies $v(t)=w(t)$ for all $t\in[0,T]$.

Existence: Let $\{\omega^k\}$ be a basis in $W_0^{1,2}(\Omega)$ consisting of eigenvectors of the Laplace operator[‡] for which λ_k are the corresponding eigenvalues. This means that

$$(\nabla\omega^k,\nabla\varphi)=\lambda_k(\omega^k,\varphi) \qquad \forall\,\varphi\in W_0^{1,2}(\Omega).$$

If $v(t)$ is an element of $W_0^{1,2}(\Omega)$ then there exist $\gamma_j(t)\in\mathbb{R}$ such

[‡] See Remark 4.14 in the Appendix for some comments on the existence of such a basis. Let us recall two advantages of this construction. Firstly, $\{\omega^k\}$ are orthogonal both in $L^2(\Omega)$ and in $W_0^{1,2}(\Omega)$ and they can be orthonormalized either in $W_0^{1,2}(\Omega)$ or in $L^2(\Omega)$. In our case, we assume that the basis is orthonormal in $L^2(\Omega)$. Secondly, $\omega^k\in C^\infty(\overline{\Omega})$ if $\partial\Omega$ is smooth enough, which is a consequence of the regularity theory for elliptic equations.

that $v(t) = \sum_{j=1}^{\infty} \gamma_j(t)\omega^j$. Denoting $v^m(t) = \sum_{j=1}^{m} \gamma_j(t)\omega^j$, we can define the mapping $\mathbb{P}_m$ by

$$\mathbb{P}_m v = v^m \, .$$

Then due to the properties of $\{\omega^k\}$, $\mathbb{P}_m$ is a continuous orthogonal projector both in

$$W_0^{1,2}(\Omega) \to \operatorname{span}\{\omega^1, \omega^2, \dots, \omega^m\},$$

and in

$$L^2(\Omega) \to \operatorname{span}\{\omega^1, \omega^2, \dots, \omega^m\}.$$

Moreover,

$$\|\mathbb{P}_m v\|_{1,2} \le \|v\|_{1,2} \quad \text{and} \quad \|\mathbb{P}_m v\|_2 \le \|v\|_2 \, .$$

Let us look for coefficients $\gamma_j^m : I \to \mathbb{R}$ such that the so-called **Galerkin approximations** u^m in the form

$$u^m(t,x) = \sum_{j=1}^{m} \gamma_j^m(t)\omega^j(x)$$

solve the system of ordinary differential equations

$$\begin{aligned} \Big(\frac{\partial^2 u^m}{\partial t^2}, \omega^j\Big) + \varepsilon\Big(\nabla \frac{\partial u^m}{\partial t}, \nabla\omega^j\Big) + \int_\Omega a_i(\nabla u^m)\frac{\partial \omega^j}{\partial x_i}\, dx& \\ = (f, \omega^j)\, , \quad j = 1, \dots, m\, ,& \end{aligned} \tag{3.49}$$

$$u^m(0,x) = \mathbb{P}_m u_0\, , \quad \frac{\partial u^m}{\partial t}(0,x) = \mathbb{P}_m u_1\, . \tag{3.50}$$

Then the system (3.49), called the **Galerkin system**, can be rewritten in the form (we write $\gamma_j = \gamma_j^m$ for simplicity)

$$\begin{aligned} \frac{d^2}{dt^2}\gamma_j(t) + \varepsilon\,\lambda_j \frac{d}{dt}\gamma_j(t) = (f, \omega^j) + \int_\Omega a_i(\nabla u^m)\frac{\partial \omega^j}{\partial x_i}\, dx\, ,& \\ \gamma_j(0) = (\mathbb{P}_m(u_0))_j\, , \quad \frac{d}{dt}\gamma_j(0) = (\mathbb{P}_m(u_1))_j\, ,& \end{aligned} \tag{3.51}$$

where $j = 1, \dots, m$. If we put $z_j(t) \equiv \frac{d}{dt}\gamma_j(t)$ and denote $\mathbf{z} \equiv (z_1, \dots, z_m)$, $\boldsymbol{\lambda} \equiv (\lambda_1, \dots, \lambda_m)$ and $\boldsymbol{\gamma} \equiv (\gamma_1, \dots, \gamma_m)$, then (3.51) takes the form

$$\begin{aligned} \frac{d}{dt}\mathbf{z} + \varepsilon\,\boldsymbol{\lambda}\cdot\mathbf{z} = \mathbf{F}(\boldsymbol{\gamma}, f, \mathbf{a})\, ,& \\ \frac{d}{dt}\boldsymbol{\gamma} = \mathbf{z}\, ,& \end{aligned} \tag{3.52}$$

or for $\mathbf{y} \equiv (\mathbf{z}, \boldsymbol{\gamma})^T$ and $\mathbf{H} \equiv (\mathbf{F} - \varepsilon\, \boldsymbol{\lambda} \cdot \mathbf{z}, \mathbf{z})^T$,

$$\begin{aligned} &\frac{d}{dt}\mathbf{y} = \mathbf{H}(\mathbf{y})\,, \\ &\mathbf{y}(0) = \mathbf{y}_0\,. \end{aligned} \tag{3.53}$$

Since the right-hand side of (3.53) satisfies the Carathéodory conditions, the existence of a solution is ensured due to Theorem 3.4 in the Appendix.

The global existence then follows from the following *a priori* estimates, namely from (3.54) and (3.55). These estimates give uniform bounds on $\mathbf{y}$ on the interval of local existence and allow to prolong the local solution onto the whole interval I.

For all $t \in (0, T]$ we have:

$$\|u^m\|_{L^\infty(0,t;W_0^{1,2}(\Omega))} \le C\,, \tag{3.54}$$

$$\Big\|\frac{\partial u^m}{\partial t}\Big\|_{L^\infty(0,t;L^2(\Omega))} \le C\,, \tag{3.55}$$

$$\varepsilon \int_0^t \Big\|\nabla \frac{\partial u^m}{\partial t}(\tau)\Big\|_2^2 \, d\tau \le C\,, \tag{3.56}$$

$$\Big\|\frac{\partial^2 u^m}{\partial t^2}\Big\|_{L^2(0,t;W^{-1,2}(\Omega))} \le C\,, \tag{3.57}$$

where C is independent of m and ε.

Let us first derive (3.54)–(3.56). Multiplying the jth equation of (3.49) by $\frac{d}{dt}\gamma_j^m(t)$ and adding the resulting equations, we obtain

$$\begin{aligned} \Big(\frac{\partial^2 u^m}{\partial t^2}, \frac{\partial u^m}{\partial t}\Big) &+ \varepsilon \Big(\nabla \frac{\partial u^m}{\partial t}, \nabla \frac{\partial u^m}{\partial t}\Big) \\ &+ \int_\Omega a_i(\nabla u^m) \frac{\partial}{\partial x_i} \frac{\partial u^m}{\partial t}\, dx = \Big(f, \frac{\partial u^m}{\partial t}\Big)\,. \end{aligned} \tag{3.58}$$

Using (3.4), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big\|\frac{\partial u^m}{\partial t}\Big\|_2^2 + \varepsilon \Big\|\nabla \frac{\partial u^m}{\partial t}\Big\|_2^2 + \frac{d}{dt}\int_\Omega \vartheta(\nabla u^m)\, dx \\ = \Big(f, \frac{\partial u^m}{\partial t}\Big)\,. \end{aligned} \tag{3.59}$$

Let us integrate (3.59) over $(0, t)$, where $t \in (0, T]$. Applying

Hölder's inequality to the right-hand side, we obtain

$$\begin{aligned}&\frac{1}{2}\Big\|\frac{\partial u^m}{\partial t}(t)\Big\|_2^2+\varepsilon\int_0^t\Big\|\nabla\frac{\partial u^m}{\partial t}(s)\Big\|_2^2\,ds+\int_\Omega\vartheta(\nabla u^m(t))\,dx\\ &\le\frac{1}{2}\Big\|\frac{\partial u^m}{\partial t}(0)\Big\|_2^2+\int_\Omega\vartheta(\nabla u^m(0))\,dx+\int_0^t\|f(\tau)\|_2\Big\|\frac{\partial u^m}{\partial t}(\tau)\Big\|_2\,.\end{aligned}$$

Using (3.9), Young's inequality and the properties of $\mathbb{P}_m$, we obtain

$$\begin{aligned}&\frac{1}{2}\Big\|\frac{\partial u^m}{\partial t}(t)\Big\|_2^2+\varepsilon\int_0^t\Big\|\nabla\frac{\partial u^m}{\partial t}(s)\Big\|_2^2\,ds+\frac{\alpha}{2}\|\nabla u^m(t)\|_2^2\\ &\quad\le\frac{1}{2}c(u_0,u_1,f)+\frac{1}{2}\int_0^t\Big\|\frac{\partial u^m}{\partial t}(\tau)\Big\|_2^2,\end{aligned}\tag{3.60}$$

where $c(u_0,u_1,f)\equiv\|u_1\|_2^2+\beta\|\nabla u_0\|_2^2+\int_0^t\|f(\tau)\|_2^2\,d\tau$. In particular we have

$$\Big\|\frac{\partial u^m}{\partial t}(t)\Big\|_2^2\le c(u_0,u_1,f)+\int_0^t\Big\|\frac{\partial u^m}{\partial t}(\tau)\Big\|_2^2.\tag{3.61}$$

Applying Gronwall's lemma 3.5 from the Appendix to (3.61), we obtain

$$\Big\|\frac{\partial u^m}{\partial t}(t)\Big\|_2^2\le c(u_0,u_1,f,T)\qquad\forall\,t\le T\,,\tag{3.62}$$

which is (3.55). Using again the inequality (3.60) together with (3.62) we immediately obtain (3.56) and (3.54), where C depends only on f,u_0,u_1 and T.

In order to obtain (3.57), it is enough to verify that

$$\left|\int_0^t\Big(\frac{\partial^2 u^m}{\partial t^2}(\tau),\varphi(\tau)\Big)\,d\tau\right|\le C$$

for all $\varphi\in L^2(0,t;W_0^{1,2}(\Omega))$ satisfying

$$\|\varphi\|_{L^2(0,t;W_0^{1,2}(\Omega))}\le 1\,.\tag{3.63}$$

Now,

$$\int_0^t \Big(\frac{\partial^2 u^m}{\partial t^2}, \varphi\Big)\, ds = \int_0^t \Big(\mathbb{P}_m \frac{\partial^2 u^m}{\partial t^2}, \varphi\Big)\, ds = \int_0^t \Big(\frac{\partial^2 u^m}{\partial t^2}, \mathbb{P}_m\varphi\Big)\, ds$$
$$\overset{(3.49)}{=} -\varepsilon \int_0^t \Big(\nabla \frac{\partial u^m}{\partial t}, \nabla \mathbb{P}_m \varphi\Big)\, ds + \int_0^t (f, \mathbb{P}_m\varphi)\, ds$$
$$- \int_0^t \Big(a_i(\nabla u^m), \mathbb{P}_m \frac{\partial \varphi}{\partial x_i}\Big)\, ds\,.$$

Using (3.8), (3.63) and Hölder's inequality we obtain

$$\Big|\int_0^t \Big(\frac{\partial^2 u^m}{\partial t^2}, \varphi\Big)\, ds\Big| \le \varepsilon \int_0^t \big\|\nabla \frac{\partial u^m}{\partial t}\big\|_2^2\, ds + \|f\|_{L^2(Q_T)} + c\|u^m\|^2_{L^2(0,t;W_0^{1,2}(\Omega))}\,, \tag{3.64}$$

where we used the continuity of the projector $\mathbb{P}_m$. Since all terms on the right-hand side of (3.64) are bounded (uniformly with respect to m and ε) due to (3.54)–(3.56), we obtain (3.57).

Therefore, there exists a subsequence $\{u^\mu\} \subset \{u^m\}$ such that

$$u^\mu \rightharpoonup u \quad \text{weakly in } L^2(I; W_0^{1,2}(\Omega))\,, \tag{3.65}$$
$$\frac{\partial u^\mu}{\partial t} \overset{*}{\rightharpoonup} z_2 \quad \text{weakly-}^* \text{ in } L^\infty(I; L^2(\Omega))\,, \tag{3.66}$$
$$\frac{\partial u^\mu}{\partial t} \rightharpoonup z_1 \quad \text{weakly in } L^2(I; W_0^{1,2}(\Omega))\,, \tag{3.67}$$
$$\frac{\partial^2 u^\mu}{\partial t^2} \rightharpoonup \omega \quad \text{weakly in } L^2(I; W^{-1,2}(\Omega))\,. \tag{3.68}$$

Since $C^\infty(\Omega)$ is dense both in $W^{1,2}(\Omega)$ and in $L^2(\Omega)$, we have $z_1 = z_2 \equiv z$. Now, from the formula (2.46) in Chapter 1 we obtain for all $\varphi \in C_C(I; W_0^{1,2}(\Omega))$

$$\int_0^t \Big(\frac{\partial u^\mu}{\partial t}(s), \varphi\Big)\, ds = -\int_0^t \Big(u^\mu, \frac{\partial \varphi}{\partial t}\Big)\, dt\,. \tag{3.69}$$

Letting $\mu \to \infty$ in (3.69), we obtain

$$\int_0^t (z, \varphi)\, ds = -\int_0^t \Big(u, \frac{\partial \varphi}{\partial t}\Big)\, dt\,, \tag{3.70}$$

and therefore, $z = \frac{\partial u}{\partial t}$. Analogously, $\omega = \frac{\partial^2 u}{\partial t^2}$.

Next, we want to prove that u is a weak solution to the problem (3.20)–(3.22), (3.4)–(3.7). Immediately from the definition of weak convergence it follows that for $\varphi \in L^2(I; W_0^{1,2}(\Omega))$

$$\int_0^t \Big(\frac{\partial^2 u^\mu}{\partial t^2}, \varphi\Big)\, ds \to \int_0^t \Big(\frac{\partial^2 u}{\partial t^2}, \varphi\Big)\, ds \tag{3.71}$$

and

$$\varepsilon \int_0^t \Big(\nabla \frac{\partial u^\mu}{\partial t}, \nabla\varphi\Big)\, ds \to \varepsilon \int_0^t \Big(\nabla \frac{\partial u}{\partial t}, \nabla\varphi\Big)\, ds\,. \tag{3.72}$$

It remains to show

$$\int_0^t \Big(a_i(\nabla u^\mu), \frac{\partial \varphi}{\partial x_i}\Big)\, ds \to \int_0^t \Big(a_i(\nabla u), \frac{\partial \varphi}{\partial x_i}\Big)\, ds \tag{3.73}$$

for all $\varphi \in L^2(I; W_0^{1,2}(\Omega))$. For this purpose we will prove the strong convergence

$$\nabla u^\mu \to \nabla u \text{ in } L^2(Q_T)\,. \tag{3.74}$$

In fact, we will show that

$$\nabla \frac{\partial u^\mu}{\partial t} \to \nabla \frac{\partial u}{\partial t} \text{ in } L^2(Q_T)\,, \tag{3.75}$$

which implies (3.74) due to (2.47) in Chapter 1. Indeed,

$$\begin{aligned} \|\nabla(u^\mu - u)(t)\|_2^2 = {} & \|\nabla(u^\mu - u)(0)\|_2^2 \\ & + 2\int_0^t \Big(\nabla\big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\big), \nabla\big(u^\mu - u\big)\Big)\, ds\,, \end{aligned} \tag{3.76}$$

and the right-hand side of (3.76) tends to 0 because of assumptions (3.75) and (3.65).

Let $\mathbb{L}_m : L^2(I; W^{1,2}(\Omega)) \longrightarrow L^2(I; \operatorname{span}\{\omega^1, \dots, \omega^m\})$ be a continuous projector. Then for $\varphi \in L^2(I; W_0^{1,2}(\Omega))$

$$\int_0^T \|\mathbb{L}_m\varphi - \varphi\|_{1,2}^2\, ds \to 0 \qquad \text{as } m \to \infty\,, \tag{3.77}$$

and the following equality holds (compare with (3.49)):

$$\begin{aligned} & \int_0^t \Big(\frac{\partial^2 u^\mu}{\partial t^2}, \mathbb{L}_\mu\varphi\Big)\, ds + \int_0^t \Big(a_i(\nabla u^\mu), \frac{\partial \mathbb{L}_\mu\varphi}{\partial x_i}\Big)\, ds \\ & + \varepsilon \int_0^t \Big(\nabla \frac{\partial u^\mu}{\partial t}, \nabla \mathbb{L}_\mu\varphi\Big)\, ds = \int_0^t (f, \mathbb{L}_\mu\varphi)\, ds. \end{aligned} \tag{3.78}$$

We will use $\varphi = \frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}$ as a test function in (3.78). To make the argument more transparent, let us consider each term in (3.78) separately, starting with the third one.

We have

$$\begin{aligned}
\varepsilon \int_0^t \Big(\nabla \frac{\partial u^\mu}{\partial t}, \nabla \frac{\partial u^\mu}{\partial t} - \nabla \mathbb{L}_\mu \frac{\partial u}{\partial t}\Big)\, ds \\
&= \varepsilon \int_0^t \Big(\nabla \frac{\partial u^\mu}{\partial t} - \nabla \frac{\partial u}{\partial t}, \nabla \frac{\partial u^\mu}{\partial t} - \nabla \frac{\partial u}{\partial t}\Big)\, ds \\
&\quad + \varepsilon \int_0^t \Big(\nabla \frac{\partial u}{\partial t}, \nabla \frac{\partial u^\mu}{\partial t} - \nabla \frac{\partial u}{\partial t}\Big)\, ds \qquad (3.79) \\
&\quad + \varepsilon \int_0^t \Big(\nabla \frac{\partial u^\mu}{\partial t}, \nabla \frac{\partial u}{\partial t} - \nabla \mathbb{L}_\mu \frac{\partial u}{\partial t}\Big)\, ds \\
&= \varepsilon \int_0^t \Big\| \nabla \Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\Big\|_2^2\, ds + \delta_\mu^1 + \delta_\mu^2 \,.
\end{aligned}$$

We see from (3.67) that $\delta_\mu^1 \to 0$ as $\mu \to \infty$. Also, $\delta_\mu^2 \to 0$, since $\frac{\partial u}{\partial t} \in L^2(I; W_0^{1,2}(\Omega))$ and (3.77) and (3.56) hold.

Similarly we will handle the first integral in (3.78). We obtain

$$\begin{aligned}
\int_0^t \Big(\frac{\partial^2 u^\mu}{\partial t^2}, \frac{\partial u^\mu}{\partial t} - \mathbb{L}_\mu \frac{\partial u}{\partial t}\Big)\, ds &= \int_0^t \Big(\frac{\partial^2 u^\mu}{\partial t^2} - \frac{\partial^2 u}{\partial t^2}, \frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\, ds \\
&\quad + \int_0^t \Big(\frac{\partial^2 u}{\partial t^2}, \frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\, ds \\
&\quad + \int_0^t \Big(\frac{\partial^2 u^\mu}{\partial t^2}, \frac{\partial u}{\partial t} - \mathbb{L}_\mu \frac{\partial u}{\partial t}\Big)\, ds \\
= \frac{1}{2}\Big\| \frac{\partial u^\mu}{\partial t}(t) - \frac{\partial u}{\partial t}(t)\Big\|_2^2 &- \frac{1}{2}\Big\| \frac{\partial u^\mu}{\partial t}(0) - \frac{\partial u}{\partial t}(0)\Big\|_2^2 \\
&\quad + \delta_\mu^3 + \delta_\mu^4 \,,
\end{aligned}$$

where $\delta_\mu^3 \to 0$ due to (3.67) and $\delta_\mu^4 \to 0$ due to (3.77) and (3.57). Putting

$$\delta_\mu^5 \equiv \frac{1}{2}\Big\| \frac{\partial u^\mu(0)}{\partial t} - \frac{\partial u^(0)}{\partial t}\Big\|_2^2 \,,$$

we see that $\delta_\mu^5 \to 0$ as $\mu \to \infty$.

Further, the second term in (3.78) can be rewritten as follows:

$$\begin{aligned}
&\int_0^t \Big(a_i(\nabla u^\mu), \frac{\partial}{\partial x_i}\Big(\frac{\partial u^\mu}{\partial t} - \mathbb{L}_\mu \frac{\partial u}{\partial t}\Big)\Big)\, ds \\
&\quad = \int_0^t \Big(a_i(\nabla u^\mu) - a_i(\nabla u), \frac{\partial}{\partial x_i}\Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\Big)\, ds \\
&\qquad + \int_0^t \Big(a_i(\nabla u), \frac{\partial}{\partial x_i}\Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\Big)\, ds \\
&\qquad + \int_0^t \Big(a_i(\nabla u^\mu), \frac{\partial}{\partial x_i}\Big(\frac{\partial u}{\partial t} - \mathbb{L}_\mu \frac{\partial u}{\partial t}\Big)\Big)\, ds \\
&\quad \equiv I_2 + \delta_\mu^6 + \delta_\mu^7\,.
\end{aligned}$$

From condition (3.8) it follows that $a_i(\nabla u) \in L^2(Q_T)$. This together with (3.67) implies that $\delta_\mu^6 \to 0$. Similarly, $\delta_\mu^7 \to 0$ because of (3.8), (3.54) and (3.77).

We will estimate I_2 by means of (3.10), Hölder's and Young's inequalities. We obtain

$$\begin{aligned}
|I_2| \leq & K(\varepsilon) \int_0^t \|\nabla(u^\mu - u)\|_2^2\, ds \\
& + \frac{\varepsilon}{2} \int_0^t \Big\|\nabla\Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\Big\|_2^2\, ds\,.
\end{aligned} \tag{3.80}$$

Using the formula

$$\nabla(u^\mu - u)(s) = \nabla(u^\mu - u)(0) + \int_0^s \nabla \frac{\partial}{\partial t}(u^\mu - u)(\tau)\, d\tau\,,$$

we obtain

$$\begin{aligned}
\|\nabla(u^\mu - u)(s)\|_2^2 \leq\ & 2\|\nabla u^\mu(0) - \nabla u(0)\|_2^2 \\
& + 2T \int_0^s \Big\|\nabla\Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)(\tau)\Big\|_2^2\, d\tau\,.
\end{aligned}$$

Denoting $\delta_\mu^8 \equiv 2TK(\varepsilon)\|\nabla u^\mu(0) - \nabla u(0)\|_2^2$, which clearly converges to 0 as $\mu \to \infty$, we obtain from (3.80)

$$\begin{aligned}
|I_2| \leq\ & \delta_\mu^8 + 2TK(\varepsilon) \int_0^t \int_0^s \Big\|\nabla\Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)(\tau)\Big\|_2^2\, d\tau\, ds \\
& + \frac{\varepsilon}{2} \int_0^t \Big\|\Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\Big\|_2^2\, ds\,.
\end{aligned}$$

Finally,

$$\delta_\mu^9 \equiv \int_0^t \Big(f, \frac{\partial u^\mu}{\partial t} - \mathbb{L}_\mu \frac{\partial u}{\partial t}\Big)\, ds$$

also converges to zero due to (3.77).

Let us put $\delta_\mu \equiv \sum_{j=1}^9 \delta_\mu^j$. Collecting the previous calculations, we obtain from (3.78)

$$\frac{\varepsilon}{2}\int_0^t \|\nabla\Big(\frac{\partial u^\mu}{\partial t} - \frac{\partial u}{\partial t}\Big)\|_2^2\, ds$$
$$\leq \delta_\mu + c\int_0^t \left(\int_0^t \|\nabla\Big(\frac{\partial u^\mu}{\partial \tau} - \frac{\partial u}{\partial t}\Big)\|_2^2\, ds\right)\, d\tau\,.$$

Since $\delta_\mu \to 0$ when $\mu \to \infty$, the assertion (3.75) is a consequence of Gronwall's lemma 3.5 in the Appendix.

Thus (3.75) and consequently (3.74) is proved. This means that $\nabla u^\mu \to \nabla u$ almost everywhere and also $a_i(\nabla u^\mu) \to a_i(\nabla u)$ almost everywhere, at least for a subsequence. Further, we know that for all measurable $H \subset Q_T$,

$$\int_H |a_i(\nabla u^\mu)|\, dx\, dt \overset{(3.8)}{\leq} c\int_H |\nabla u^\mu|\, dx\, dt \leq c\|\nabla u^\mu\|_{L^2(Q_T)}\sqrt{|H|}\,,$$

which implies the continuity of $a_i(\nabla u^\mu)$ with respect to the Lebesgue measure. Thus we are in position to apply Vitali's lemma 2.11 in Chapter 1 and we obtain (3.73) for all $\varphi \in L^2(0,T;W_0^{1,2}(\Omega))$. In this way, u satisfies (3.27). Moreover, since u is a weak limit of the sequence u^μ which fulfills the estimates (3.54)–(3.57), u satisfies the same estimates. Therefore, (3.30)–(3.33) hold and the proof of Theorem 3.29 is complete. ■