COMPACT SETS OF CONTINUITY FOR BOREL FUNCTIONS

VÁCLAV VLASÁK AND MIROSLAV ZELENÝ

Abstract. We investigate connections between complexity of a function $f$ from a Polish space $X$ to a Polish space $Y$ and complexity of the set $C(f) = \{ K \in \mathcal{K}(X); f|_K \text{ is continuous} \}$, where $\mathcal{K}(X)$ denotes the space of all compact subsets of $X$ equipped with the Vietoris topology. We prove that if $C(f)$ is analytic, then $f$ is Borel; and assuming $\Delta^1_2$-determinacy we show that $f$ is Borel if and only if $C(f)$ is coanalytic. Similar results for projective classes are also presented.

1. Introduction

Let $X$ be a Polish space. Denote the space of all compact subsets of $X$, which is equipped with the Vietoris topology, by $\mathcal{K}(X)$. At least since the important paper [4] it is well known that descriptive properties of families of compact sets and their set structure (like being $\sigma$-ideal or ideal) can interact in a nontrivial way (see [5] for a recent survey). These general results were applied in different parts of analysis, mainly in the theory of exceptional sets in Harmonic Analysis.

F. Jordan ([1, 2]) studies the following situation. Let $f$ be a function from a Polish space $X$ to a Polish space $Y$ and

$$C(f) := \{ K \in \mathcal{K}(X); f|_K \text{ is continuous} \}.$$ 

Jordan investigates relationships between descriptive properties of the function $f$ and the ideal $C(f)$. Besides other results he showed that, if $f$ is Borel, then $f$ is a Baire class one function provided $C(f)$ is an $F_{\sigma\delta}$ subset of $\mathcal{K}(X)$. He also proved that if $f$ is Borel and $C(f)$ is analytic, then $C(f)$ is Borel. In this note we show that the assumption of Borelness of $f$ can be omitted by proving the following result.

Theorem 1.1. Let $X,Y$ be Polish spaces and $f : X \rightarrow Y$ be a function. If $C(f)$ is an analytic subset of $\mathcal{K}(X)$, then $f$ is Borel.

One can show that if $f$ is Borel, then $C(f)$ is coanalytic (see Theorem 2.5(i)). This and Theorem 1.1 imply the next corollary giving a restriction on complexity of ideals of compact sets of the form $C(f)$.

Corollary 1.2. Let $X,Y$ be Polish spaces and $f : X \rightarrow Y$ be a function. If the ideal $C(f)$ is an analytic subset of $\mathcal{K}(X)$, then $C(f)$ is Borel.

Compare this result with those of [4], which are of a similar nature.

Further we show that, assuming $\Delta^1_2$-determinacy, Borelness of $f$ can be actually characterized by descriptive properties of $C(f)$.

2000 Mathematics Subject Classification. 03E15, 28A05, 54H05.

The work is a part of the research project MSM 0021620839 financed by MSMT and partly supported by the grant GA CR 201/06/0198.
Theorem 1.3. (Det($\Delta^1_2$)) Let $X, Y$ be Polish spaces and $f : X \to Y$ be a function. Then $f$ is Borel if and only if $C(f)$ is a coanalytic subset of $K(X)$.

Remark. We do not know whether the assumption on determinacy of $\Delta^1_2$ games can be omitted in Theorem 1.3.

The next section contains more detailed versions of our results. We formulate them also for projective classes. Proofs are given in the last section. Throughout the paper we follow the notation used in [3], where one can also find all needed definitions.

2. Results

A connection between complexities of $C(f)$ and graph $f$ is established by the following result.

Theorem 2.1. Let $X, Y$ be Polish spaces, $f : X \to Y$ be a function, and $\Gamma$ be a class of subsets of Polish spaces which is closed under Borel preimages. If $C(f)$ is $(\Sigma^1_n)$, then $K(\text{graph } f) \in \Gamma$ and graph $f \in \Gamma$.

The next corollary immediately follows from Theorem 2.1. Note that it is easy to see that $f$ is $\Delta^1_n$-measurable if and only if graph $f \in \Sigma^1_n$.

Corollary 2.2. Let $X, Y$ be Polish spaces, $n \geq 1$, and $f : X \to Y$ be a function. If $C(f)$ is $\Pi^1_n$ (respectively), then graph $f$ is $\Pi^1_n$ ($\Sigma^1_n$ respectively). In the latter case $f$ is $\Delta^1_n$-measurable.

More general forms of Theorem 1.1 and Corollary 1.2 read as follows. Theorem 2.3 was proved by Jordan ([1]) using an additional assumption that $f$ is Borel.

Theorem 2.3. Let $X, Y$ be Polish spaces and $f : X \to Y$ be a function. Then the following are equivalent:

(i) $C(f)$ is Borel,

(ii) $C(f)$ is analytic,

(iii) $f$ has a $G_\delta$ graph.

Theorem 2.4. Let $X, Y$ be Polish spaces and $f : X \to Y$ be a function. If $C(f)$ is $\Sigma^1_n$, then $C(f)$ is $\Delta^1_n$.

The following theorem provides a characterization of $\Delta^1_n$-measurable functions (the assertions (i) and (ii)) and of functions having $\Pi^1_n$ graph ((iii) and (iv)) assuming $\text{Det}(\Delta^1_{n+1})$.

Theorem 2.5. Let $X, Y$ be Polish spaces, $n \geq 1$, and $f : X \to Y$ be a function.

(i) If $f$ is $\Delta^1_n$-measurable, then $C(f)$ is $\Pi^1_n$.

(ii) (Det($\Delta^1_{n+1}$)) If $C(f)$ is $\Pi^1_n$, then $f$ is $\Delta^1_n$-measurable.

(iii) If $f$ has $\Pi^1_n$ graph, then $C(f)$ is $\Delta^1_{n+1}$.

(iv) (Det($\Delta^1_{n+1}$)) If $C(f)$ is $\Delta^1_{n+1}$, then $f$ has a $\Pi^1_n$ graph.

Corollary 2.6. Let $X, Y$ be Polish spaces and $f : X \to Y$ be a function. Then $f$ is a projective function if and only if $C(f)$ is projective.
3. Proofs

3.1. Notation. Let $X$ and $Y$ be Polish spaces and $f : X \to Y$ be a continuous function. Then the function $\hat{f} : \mathcal{K}(X) \to \mathcal{K}(Y)$ is defined by $\hat{f}(K) = f(K)$. If $A \subset X$, then $\mathcal{K}(A)$ stands for the set of all compact subsets of $A$. The symbols $\pi_X$ and $\pi_Y$ denote the projections from $X \times Y$ to $X$ and to $Y$ respectively. If $x \in X$, then $U(x)$ denotes the family of all open neighborhoods of $x$.

3.2. Proof of Theorem 2.1.

Lemma 3.1. Let $X,Y$ be Polish spaces and $f : X \to Y$ be a function. Then $C(f) = \pi_X(\mathcal{K}(\text{graph } f))$.

Proof. Define $\Psi : X \to X \times Y$ by $\Psi(x) = (x,f(x))$. Let $K \in C(f)$. Then $\Psi$ is continuous on $K$. Consequently, $\Psi(K) \subset \text{graph } f$ is compact. So, $K \in \pi_X(\mathcal{K}(\text{graph } f))$.

Let $K \in \pi_X(\mathcal{K}(\text{graph } f))$ be arbitrary. Then $\text{graph}(f|_K)$ is compact and $f|_K$ is clearly continuous. □

Lemma 3.2. Let $X$ be a Polish space and $D$ be a countable dense subset of $X$. Then there exists a Borel function $\Phi : \mathcal{K}(X) \to \mathcal{K}(X)$ such that for every $K \in \mathcal{K}(X)$

\begin{itemize}
  \item $K \subset \Phi(K) \subset K \cup D$,
  \item $\Phi(K) \cap D = \Phi(K)$.
\end{itemize}

Proof. Let $\rho$ be a compatible complete metric on $X$ with $\rho \leq 1$. Let $d_H$ be the corresponding Hausdorff metric on $\mathcal{K}(X)$. Let $(F_i)_{i \in \omega}$ be a sequence of all finite subsets of $D$. For every $n \in \omega$ we define an auxiliary function $\varphi_n : \mathcal{K}(X) \to \mathcal{K}(X)$ as follows. The value $\varphi_n(K)$ equals $F_j$ where $j \in \omega$ is the smallest number with $d_H(K,F_j) < \frac{1}{n+1}$. Since $\{F_i; \ i \in \omega\}$ is dense in $\mathcal{K}(X)$, the definition is correct and $\lim_{n \to \infty} \varphi_n(K) = K$ for every $K \in \mathcal{K}(X)$. Thus $K \cup \bigcup_{n \in \omega} \varphi_n(K) \in \mathcal{K}(X)$.

We define $\Phi : \mathcal{K}(X) \to \mathcal{K}(X)$ by

$$
\Phi(K) = K \cup \bigcup_{n \in \omega} \varphi_n(K).
$$

Let $V \subset X$ be open. Since $\varphi_n$ is obviously Borel for each $n \in \omega$ and $\Phi(K) \cap V \neq \emptyset$ if and only if $K \cap V \neq \emptyset$ or $\exists n \in \omega : \varphi_n(K) \cap V \neq \emptyset$, we see that the set $\{K \in \mathcal{K}(X) ; \Phi(K) \cap V \neq \emptyset\}$ is Borel. Now it is easy to infer that $\Phi$ is a Borel function.

Let $K \in \mathcal{K}(X)$. Then $K \subset \Phi(K) \subset K \cup D$ by definition. Further, we have

$$
\Phi(K) \subset \bigcup_{n \in \omega} \varphi_n(K) \subset \Phi(K) \cap D \subset \Phi(K)
$$

and we are done. □

Proof of Theorem 2.1. Find a set $D \subset \text{graph } f$ which is countable and dense in graph $f$. Let $\Phi : \mathcal{K}(D) \to \mathcal{K}(D)$ be the function from Lemma 3.2, where $X$ is replaced by $D$. We show

\begin{equation}
(\pi_X \circ \Phi)^{-1}(C(f)) = \mathcal{K}(\text{graph } f).
\end{equation}
Let $K \in \mathcal{K}(\text{graph } f)$ be arbitrary. Since $\Phi(K) \subset K \cup D \subset \text{graph } f$, we have $\Phi(K) \in \mathcal{K}(\text{graph } f)$. By Lemma 3.1, $\pi_X(\Phi(K)) \in C(f)$ and therefore $K \in (\pi_X \circ \Phi)^{-1}(C(f))$.

Let $K \in (\pi_X \circ \Phi)^{-1}(C(f))$. The graph of $f \upharpoonright \pi_X(\Phi(K))$ is compact and $\Phi(K) \cap D \subset \text{graph}(f \upharpoonright \pi_X(\Phi(K)))$. Then

$$K \subset \Phi(K) = \Phi(K) \cap D \subset \text{graph}(f \upharpoonright \pi_X(\Phi(K))) = \text{graph}(f \upharpoonright \pi_X(\Phi(K))) \subset \text{graph } f.$$ 

Thus, $K \in \mathcal{K}(\text{graph } f)$ and (3.1) is proved.

Let $C(f) \in \Gamma$. The formula (3.1) implies that $\mathcal{K}(\text{graph } f) \in \Gamma$. Further, the function $S : X \to \mathcal{K}(X)$ defined by $x \mapsto \{x\}$ is continuous. Then we have $\text{graph } f = S^{-1}(\mathcal{K}(\text{graph } f))$. This implies graph $f \in \Gamma$. \hfill \Box

### 3.3. Proof of Theorem 2.3.

(iii) $\Rightarrow$ (i) According to the well-known fact the set $\mathcal{K}(\text{graph } f)$ is also $G_δ$. The function $\pi_X \upharpoonright \mathcal{K}(\text{graph } f)$ is injective and by Lemma 3.1 we have $C(f) = \pi_X(\mathcal{K}(\text{graph } f))$. Thus $C(f)$ is Borel.

(ii) $\Rightarrow$ (iii) The set $\mathcal{K}(\text{graph } f)$ is clearly a $\sigma$-ideal and it is analytic by Theorem 2.1. Theorem 11 of [4, Section 1] says that each analytic $\sigma$-ideal is in fact $G_δ$. Thus $\mathcal{K}(\text{graph } f)$ is $G_δ$ and, consequently, graph $f$ is $G_δ$ as well.

(i) $\Rightarrow$ (ii) This implication is trivial.

### 3.4. Proof of Theorem 2.4.

Assume that $C(f)$ is $\Sigma^1_2$. Then Corollary 2.2 gives that $f$ is $\Delta^1_α$-measurable. Now the fact that $\mathcal{K}(A)$ is $\Pi^1_α$ provided $A$ is $\Pi^1_α$ and the next lemma gives that $C(f)$ is $\Pi^1_α$. Thus $C(f)$ is $\Delta^1_α$.

The next lemma is inspired by [2].

**Lemma 3.3.** Let $X, Y$ be Polish spaces, $n \geq 1$, and $f : X \to Y$ a function. If $f$ is $\Delta^1_α$-measurable, then there exist sets $H^i \in \Delta^1_α(X)$, $i, l \in \omega$, such that $C(f) = \bigcap_{i \in \omega} \bigcup_{l \in \omega} \mathcal{K}(H^i)$.

**Proof.** Let $\rho$ and $\beta$ be compatible complete metrics on $X$ and $Y$ respectively. Set $A^x = \pi_Y(A \cap \{x\} \times Y)$ for $x \in X$ and $A \subset X \times Y$. Let $\mathcal{V}$ and $\mathcal{U}$ be countable bases for $X$ and $Y$ respectively containing only closed balls. Let $\mathcal{W}$ be the set of all finite collections $\mathcal{Z}$ of sets of the form $B_1 \times B_2$ with $B_1 \in \mathcal{V}$, $B_2 \in \mathcal{U}$. Further we define $\mathcal{W}_l$, $l \in \omega$, as the system of all $\mathcal{Z} \in \mathcal{W}$ such that $\text{diam}_l((\bigcup \mathcal{Z})^x) < \frac{1}{l+1}$ whenever $x \in \pi_X((\bigcup \mathcal{Z})^x)$. Let $\mathcal{W}_l = \{\mathcal{Z}^i : i \in \omega\}$. Set $H^i = \pi_X((\bigcup \mathcal{Z}^i) \cap \text{graph } f)$. We have that $H^i$ is $\Delta^1_α$ by $\Delta^1_α$-measurability of $f$. Set $T = \bigcap_{i \in \omega} \bigcup_{l \in \omega} \mathcal{K}(H^i)$. We prove $T = C(f)$.

Let $K \in T$, $x \in K$, and $\varepsilon > 0$ be arbitrary. Then there exist $l \in \omega$ and $\mathcal{Z} \in \mathcal{W}_l$ such that $\frac{1}{l+1} < \varepsilon$ and $K \subset \pi_X((\bigcup \mathcal{Z}) \cap \text{graph } f)$. Set $\mathcal{P} = \{B_1 \times B_2 : B_1 \in \mathcal{V}, B_2 \in \mathcal{U}, x \notin B_1\}$. Since $\mathcal{Z}$ is finite, one can find $\delta > 0$ with $\text{diam}_l(x, \delta) \cap \pi_X(\mathcal{P}) = \emptyset$. Let $\bar{x} \in B_\rho(x, \delta) \cap K$ be arbitrary. Then $f(\bar{x}) \in \pi_Y((\bigcup (\mathcal{Z} \setminus \mathcal{P})) = (\bigcup \mathcal{Z})^x$. Thus, $\beta(f(x), f(\bar{x})) < \varepsilon$ since $\text{diam}(\bigcup \mathcal{Z})^x < \frac{1}{l+1} < \varepsilon$. This gives $K \in C(f)$.

Let $K \in C(f)$ and $l \in \omega$ be arbitrary. Then there exists $m \in \omega$ such that for every $x, \bar{x} \in K$ with $\rho(x, \bar{x}) < \frac{2}{m}$ we have $\beta(f(x), f(\bar{x})) < \frac{1}{m+1}$. Then for every $x \in K$ there exist $B_{1,x} \in V$ and $B_{2,x} \in \mathcal{U}$ such that

(a) $x$ is in the interior of $B_{1,x}$,
(b) $\text{diam}_\rho B_{1,x} < \frac{1}{m}$,
(c) $f(B_{1,x}) \subset B_{2,x}$, and
(d) $\text{diam}_\rho B_{2,x} < \frac{1}{m+1}$. 

Clearly, the union of interiors of $B_{1,x}$, $x \in K$, covers $K$. Since $K$ is compact, we can find finitely many points $x_1, \ldots, x_m$ in $K$ such that the balls $B_{1,x_1}, \ldots, B_{1,x_m}$ cover $K$. Set

$$Z = \{B_{1,x_s} \times B_{2,x_s}; \, s = 1, \ldots, m\}.$$ 

Let $\tilde{x} \in \pi_X(\bigcup Z)$ and $y, z \in (\bigcup Z)^x$. Then there exist $1 \leq i, j \leq m$ such that $y \in B_{2,x_i}$, $z \in B_{2,x_j}$, and $\tilde{x} \in B_{1,x_i} \cap B_{1,x_j}$. This yields $\rho(x_i, x_j) \leq \frac{1}{m}$ and, consequently, $\beta(f(x_i), f(x_j)) < \frac{1}{1+4m}$. Using (c) and (d) we also have $\beta(f(x_i), y) \leq \frac{1}{1+4m}$ and $\beta(f(x_j), z) \leq \frac{1}{1+4m}$. Therefore, $\beta(y, z) < \frac{1}{1+4m}$ and, consequently, $\text{diam}_B(\bigcup Z)^x < \frac{1}{4m}$. This implies $Z \in \mathcal{M}_1$. Using (c) we get that $Z$ covers $\text{graph}(f|_K)$. Thus we have $K \in T$. □

3.5. Proof of Theorem 2.5.

Lemma 3.4. Let $X, Y$ be Polish spaces, $n \geq 1$, and $f : X \to Y$ be a function. If $f$ is not $\Delta^1_{n+1}$-measurable then there exist $x \in X$ and $U \in \mathcal{U}(f(x))$ such that, for every $U' \in \mathcal{U}(f(x))$ and $V \in \mathcal{U}(x)$, the set $f^{-1}(U') \cap V$ cannot be separated from the set $f^{-1}(Y \setminus U) \cap V$ by a $\Pi^1_n$ subset of $V$.

Proof. Let $\mathcal{V}$ and $\mathcal{U}$ be countable open bases of $X$ and $Y$ respectively. Since $f$ is not $\Delta^1_{n+1}$-measurable one can find an open set $W \subset Y$ such that $f^{-1}(W)$ is not in $\Pi^1_n$. For $U \in \mathcal{U}$ and $V \in \mathcal{V}$, let $L(U, V)$ be a $\Pi^1_n$ subset of $V$ separating $f^{-1}(U) \cap V$ from $f^{-1}(Y \setminus U) \cap V$, if such a set exists, otherwise set $L(U, V) := \emptyset$.

Suppose towards a contradiction that the desired $x$ and $U$ do not exist. Then $f^{-1}(W) = \bigcup \{L(U, V); \, U \in \mathcal{U}, V \in \mathcal{V}\}$. Since $\mathcal{U}$ and $\mathcal{V}$ are countable, we have that $f^{-1}(W)$ is $\Pi^1_n$, a contradiction. □

Lemma 3.5. (Det($\Delta^1_{n+1}$)) Let $X$ be a Polish space, $n \geq 1$, $A, B \in \Delta^1_{n+1}(X)$, and $A \cap B = \emptyset$. If there is no $\Pi^1_n$ set separating $A$ from $B$, then there is a compact set $C \subset A \cup B$ such that $C \cap A$ is $\Sigma^1_n$-hard. In particular, if $D \subset X$ is $\Delta^1_{n+1} \setminus \Pi^1_n$, then $D$ is $\Sigma^1_n$-hard.

Proof. First assume that $X = \omega^\omega$. Let $Q \subset 2^{\omega}$ be $\Sigma^1_n$-complete. Consider the separation game $SG(Q; A, B)$ as in [3, 21.F]. This game is determined by Det($\Delta^1_{n+1}$).

Since there is no $\Pi^1_n$ set separating $A$ from $B$, player I cannot have a winning strategy. So $\Pi$ has a winning strategy, which gives a compact set $C \subset A \cup B$ such that $C \cap A$ is $\Sigma^1_n$-hard (see [3, 21.F]).

Now consider the general case. Let $\varphi : \omega^\omega \to X$ be a continuous surjection. Denote $A = \varphi^{-1}(A)$ and $B = \varphi^{-1}(B)$. The sets $\tilde{A}, \tilde{B}$ are $\Delta^1_{n+1}$. Assume that $T \subset \omega^\omega$ is a $\Pi^1_n$ set separating $\tilde{A}$ from $\tilde{B}$. Then $X \setminus \varphi(\omega^\omega \setminus T)$ is a $\Pi^1_n$ set separating $A$ from $B$, a contradiction. Thus $A$ cannot be separated by a $\Pi^1_n$ set from $B$. This means that there is a compact set $\tilde{C} \subset \tilde{A} \cup \tilde{B}$ such that $\tilde{C} \cap \tilde{A}$ is $\Sigma^1_n$-hard. Setting $C := \varphi(\tilde{C})$ we are done. □

The next lemma is also inspired by [2].

Lemma 3.6. (Det($\Delta^1_{n+1}$)) Let $X, Y$ be Polish spaces, $n \geq 1$, and $f : X \to Y$ be a function. If $f$ is not $\Delta^1_{n+1}$-measurable and $f$ is $\Delta^1_{n+1}$-measurable, then $C(f)$ is $\Sigma^1_n$-hard.
Proof. Let \( \rho \) be a complete compatible metric on \( X \) and let \( \beta \) be a complete compatible metric on \( Y \). By Lemma 3.4, there are \( p \in X \), \( U \in \mathcal{U}(f(p)) \), and decreasing sequences \( (V_i) \) and \( (U_i) \) of open sets in \( X \) and \( Y \) respectively such that

- \( \lim \diam p V_i = 0 \),
- \( \lim \diam \beta U_i = 0 \)

and, for every \( l \in \omega \),

- \( p \in V_i \), \( f(p) \in U_i \subset U \),
- \( A_i := f^{-1}(U_i) \cap V_i \) cannot be separated from the set \( B_i := f^{-1}(Y \setminus U) \cap V_i \) by a \( \Pi^1_0 \) subset of \( V_i \).

Since \( A_i, B_i \in \Delta_{n+1}^1(X) \) and \( A_i \cap B_i = \emptyset \), Lemma 3.5 guarantees that there is a compact set \( K_i \subset A_i \cup B_i \subset V_i \) such that \( D_i := K_i \cap A_i \) is \( \Sigma^1_{n+1} \)-hard.

Let \( K := \{ p \} \cup \bigcup i \in \omega K_i \). The set \( K \) is clearly compact, since \( K_i \to \{ p \} \). Let \( h : \prod_{l \in \omega} K_l \to \mathcal{K}(K) \) be defined by

\[
h(\sigma) = \{ \sigma(l); l \in \omega \} \cup \{ p \}.
\]

It is easy to verify that \( h \) is well-defined, continuous, and that

\[
T := h^{-1}(C(f|_K)) = \{ \sigma \in \prod_{l \in \omega} K_l; \exists i_0 \in \omega \forall i > i_0 : \sigma(i) \in D_i \}.
\]

We show that \( T \) is \( \Sigma^1_{n+1} \)-hard. Let \( B \) be a \( \Sigma^1_0 \) subset of \( \omega^\omega \). Find a continuous function \( \psi : \omega^\omega \to K_l \) such that \( \psi^{-1}(B) = B \). Let \( \psi : \omega^\omega \to \prod_{l \in \omega} K_l \) be defined by \( \psi(\nu)(l) = \psi(l)(\nu) \). It is easy to see that \( \psi^{-1}(T) = B \) and \( \psi \) is continuous. Thus, \( C(f|_K) \) is \( \Sigma^1_{n+1} \)-hard. So, \( C(f) \) is \( \Sigma^1_{n+1} \)-hard. \( \square \)

Proof of Theorem 2.5. (i) Let \( f \) be \( \Delta^1_{n+1} \)-measurable. By Lemma 3.3 there exist sets \( H^1_i \in \Delta^1_{n+1}(X) \) such that \( C(f) = \bigcap_{i \in \omega} \bigcup_{i \in \omega} \mathcal{K}(H^1_i) \). Since \( \mathcal{K}(H^1_i) \) is \( \Pi^1_0 \), we get \( C(f) \in \Pi^1_0(\mathcal{K}(X)) \).

(ii) Let \( C(f) \) be \( \Pi^1_0 \). By Corollary 2.2 the function \( f \) is \( \Delta^1_{n+1} \)-measurable. Suppose \( f \) is not \( \Delta^1_{n+1} \)-measurable. According to Lemma 3.6, \( C(f) \) is \( \Sigma^1_{n+1} \)-hard, a contradiction.

(iii) Let \( f \) have \( \Pi^1_{n+1} \) graph. Then \( \mathcal{K}(f) \) is \( \Pi^1_{n+1} \) in \( \mathcal{K}(X \times Y) \). By Lemma 3.1 we have \( C(f) = \pi_X(\mathcal{K}(\text{graph } f)) \). This gives that \( C(f) \) is \( \Sigma^1_{n+1} \). Now Theorem 2.4 implies the desired conclusion.

(iv) Suppose that \( C(f) \) is \( \Delta^1_{n+1} \) and graph \( f \) is not \( \Pi^1_{n+1} \). Then by Theorem 2.1, graph \( f \) is in \( \Delta^1_{n+1} \setminus \Pi^1_{n+1} \). Using Lemma 3.5 we have that graph \( f \) is \( \Sigma^1_{n-1} \)-hard. Using Lemma [5, Lemma 1.1] (cf., Lemma 1 in [4, Section 1]) we have that \( \mathcal{K}(\text{graph } f) \) is \( \Pi^1_{n+1} \)-hard. On the other hand \( \mathcal{K}(\text{graph } f) \) is \( \Delta^1_{n+1} \) by Theorem 2.1. This is a contradiction. \( \square \)

References


Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75, Praha 8, Czech Republic:
E-mail address: vlasakmm@volny.cz
E-mail address: zeleny@karlin.mff.cuni.cz