ŘEŠENÍ PÍSEMKY Z 23.1.17
(var. B)

(1)
$$\lim_{x\to 0} \frac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}} = \lim_{x\to 0} \frac{x^2\left(\sqrt{1+x\sin x}+\sqrt{\cos x}\right)}{1+x\sin x-\cos x} =$$

$$= \lim_{x\to 0} \frac{\sqrt{1+x\sin x}+\sqrt{\cos x}}{\frac{1-\cos x}{x^2}+\frac{\sin x}{x}} = \frac{1+1}{\frac{1}{2}+1} = \frac{4}{3}$$

(2)
$$\int \frac{e^{3x}\,dx}{(e^x+2)^2(e^x+1)^2} = \left[\begin{array}{c} \text{subst. } e^x = t \\ e^x dx = dt \end{array}\right] =$$

$$\int \frac{t^2\,dt}{(t+2)^2(t+1)^2} = \int\left(\frac{A}{(t+2)^2}+\frac{B}{t+2}+\frac{C}{(t+1)^2}+\frac{D}{t+1}\right)dt$$

$$t^2 = A(t^2+2t+1)+B(t+2)(t^2+2t+1)+C(t^2+4t+4)$$
$$+D(t+1)(t^2+4t+4)$$

$t^3:\quad 0 = B+D \Rightarrow \boxed{B=-D}$

$t^2:\quad 1 = A+4B+C+5D$

$t^1:\quad 0 = 2A+5B+4C+8D$

$t^0:\quad 0 = A+2B+4C+4D$

$1 = A+C+D$

$0 = 2A+4C+3D$

$0 = A+4C+2D \Rightarrow A = -4C-2D$

$$\left.\begin{array}{l} 1 = -3C-D \\ 0 = -4C-D \end{array}\right\} \Rightarrow \boxed{1 = C} \Rightarrow \boxed{D=-4}$$

$$\Rightarrow \boxed{A=4} \qquad \boxed{B=4}$$

$$\int \frac{t^2}{(t+2)^2(t+1)^2}\,dt = \int\left(\frac{4}{(t+2)^2} + \frac{4}{t+2} + \frac{1}{(t+1)^2} - \frac{4}{t+1}\right)dt =$$

$$\overset{c}{=} -\frac{4}{t+2} + 4\ln|t+2| - \frac{1}{t+1} - 4\ln|t+1| =$$

$$= -\frac{4}{e^x+2} + 4\ln(e^x+2) - \frac{1}{e^x+1} - 4\ln(e^x+1)$$

$x \in \mathbb{R}$

(3) $y' + \frac{2}{x}y = \sin x$ $\quad / e^{2\ln x} = x^2$

ODR 1. řádu, lineární

$p(x) = \frac{2}{x}$

$P(x) = \int \frac{2}{x} = 2\ln x \quad (x>0)$

$e^{P(x)} = e^{2\ln x} = e^{\ln x^2} = x^2$

$$x^2y' + 2xy = x^2\sin x$$

$$(x^2y)' = x^2\sin x$$

$$x^2y = c + \int x^2\sin x \underset{2\times \text{ per partes}}{=} c - x^2\cos x + 2\cos x + 2x\sin x$$

$$\boxed{y = \frac{c}{x^2} - \cos x + \frac{2}{x^2}\cos x + \frac{2}{x}\sin x}$$

$x \in (-\infty, 0) \cup (0, +\infty)$

4. $f(x) = \sqrt{x + \frac{1}{x}}$

- Def. obor: $x + \frac{1}{x} \geq 0$ & $x \neq 0$

  pro $x > 0$: $x^2 + 1 \geq 0$ ✓

  pro $x < 0$: $x^2 + 1 \leq 0$ ✗

  $D(f) = \{x, x > 0\} = (0, \infty)$

- není sudá, lichá ani periodická

- $f \in C(0,\infty)$

- limity v kraj. bodech: $\lim_{x \to 0+} \sqrt{x + \frac{1}{x}} = +\infty$

  $\lim_{x\to\infty} \sqrt{x + \frac{1}{x}} = +\infty$

- $$f'(x) = \frac{1}{2} \cdot \frac{1 - \frac{1}{x^2}}{\sqrt{x + \frac{1}{x}}}$$

  $$= \underbrace{\frac{1}{2x^2}\sqrt{\frac{x}{x^2+1}}}_{>0}\ \underbrace{(x^2 - 1)}$$

  > 0 pro $x > 1$ roste na $(1, +\infty)$

  < 0 pro $x < 1$ klesá na $(0, 1)$

  v 1 je minimum

  $f(1) = \sqrt{2}$ $\quad H(f) = \langle\sqrt{2}, \infty)$

- $$f''(x) = \left(\frac{1}{2}\left(1 - \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right)^{-\frac{1}{2}}\right)'$$

  $$= \frac{1}{2} \cdot \frac{2}{x^3}\left(x + \frac{1}{x}\right)^{-\frac{1}{2}} + \frac{1}{2}\left(1 - \frac{1}{x^2}\right)\left(-\frac{1}{2}\right)\frac{1}{\sqrt{\left(x + \frac{1}{x}\right)^3}}\left(1 - \frac{1}{x^2}\right)$$

  $$= \frac{1}{x^3\sqrt{x + \frac{1}{x}}} - \frac{1}{4}\frac{\left(1 - \frac{1}{x^2}\right)^2}{\sqrt{\left(x + \frac{1}{x}\right)^3}} = \frac{1}{\sqrt{x + \frac{1}{x}}}\left(\frac{1}{x^3} - \frac{1}{4}\frac{\left(1 - \frac{1}{x^2}\right)^2}{x + \frac{1}{x}}\right)$$

  $$= \frac{1}{x^3\sqrt{x + \frac{1}{x}}}\left(1 - \frac{1}{4} \cdot \frac{x^4\left(1 - \frac{2}{x^2} + \frac{1}{x^4}\right)}{x^2 + 1}\right)$$

  $$= \underbrace{\frac{1}{4(x^2+1)x^3\sqrt{x + \frac{1}{x}}}}_{>0}\left(\underbrace{4x^2 + 4 - x^4 + 2x^2 - 1}_{-(x^4 - 6x^2 - 3)}\right)$$

$$f''(x) = -\frac{x^4 - 6x^2 - 3}{4(x^2+1)\,x^3\sqrt{x+\frac{1}{x}}}$$

O znaménku f'' rozhoduje $x^4 - 6x^2 - 3 = 0$

$$x^2_{1,2} = \frac{6 \pm \sqrt{36+12}}{2} = 3 \pm 2\sqrt{3}$$

$$3 - 2\sqrt{3} < 0 \Rightarrow x^2 = 3 + 2\sqrt{3}$$

$$x = \pm\sqrt{3+2\sqrt{3}}$$

Přitom $-\sqrt{3+2\sqrt{3}} < 0$ .... mimo def. obor

$\Rightarrow$ jediný bod inflexe je $x = \sqrt{3+2\sqrt{3}} \approx 2{,}54$

f je konvexní na $(0, \sqrt{3+2\sqrt{3}})$

konkávní na $(\sqrt{3+2\sqrt{3}}, +\infty)$

Náčrt :