ŘEŠENÍ PÍSEMKY Z 31.1.2017
varianta C

(1) $$\lim_{x\to 0}\left(\frac{1+x4^x}{1+x3^x}\right)^{\frac{1}{x^2}} = \text{„}1^\infty\text{"} \text{ neurčitý výraz}$$

$$= \lim_{x\to 0} \exp\left(\frac{1}{x^2}\ln\left(\frac{1+x4^x}{1+x3^x}\right)\right) = \exp\Big(\underbrace{\lim_{x\to 0}\frac{1}{x^2}\ln\left(\frac{1+x4^x}{1+x3^x}\right)}_{=:A}\Big)$$

$$A = \lim_{x\to 0}\left(\frac{1}{x^2}\underbrace{\frac{\ln\left(\frac{1+x4^x}{1+x3^x}\right)}{\frac{1+x4^x}{1+x3^x}-1}}_{\to 1}\cdot\left(\frac{1+x4^x-(1+x3^x)}{1+x3^x}\right)\right)$$

$$= \lim_{x\to 0}\frac{1}{x^2}\,\frac{x(4^x-3^x)}{1+x3^x} = \lim_{x\to 0}\underbrace{\frac{4^x-3^x}{x}}_{\text{L'Hospital}}\cdot\underbrace{\frac{1}{1+x3^x}}_{\to 1}$$

$$= \lim_{x\to 0}\left(4^x\ln 4 - 3^x\ln 3\right) = \ln\frac{4}{3}$$

tj $A = \ln\frac{4}{3}$ a celkově

je výsledná limita $= \boxed{\frac{4}{3}}$

2

$$\int \frac{x^4+x^3+x^2+x+1}{(x+1)(x^2+x+3)} = \underbrace{\int \frac{x^4+x^3+x^2+x+1}{x^3+2x^2+4x+3}}_{\text{nejprve musím dělit}}$$

$$\begin{array}{l} (x^4+x^3+x^2+x+1):(x^3+2x^2+4x+3) = x-1 \\ -(x^4+2x^3+4x^2+3x) \\ \hline \quad -x^3-3x^2-2x+1 \\ \quad -(-x^3-2x^2-4x-3) \\ \hline \quad\quad -x^2+2x+4 \end{array}$$

Parciální zlomky:

$$\frac{-x^2+2x+4}{(x+1)(x^2+x+3)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+3}$$

$$-x^2+2x+4 = A(x^2+x+3) + (Bx+C)(x+1)$$

$$\left.\begin{array}{lrcl} x^2: & -1 &=& A+B \\ x^1: & 2 &=& A+B+C \end{array}\right\} \Rightarrow C=3$$

$$x^0: \quad 4 = 3A+C$$

$$A = \frac{1}{3}$$

$$B = -\frac{4}{3}$$

Tedy

$$\int \frac{x^4+x^3+x^2+x+1}{(x+1)(x^2+x+3)} = \int(x-1) + \frac{1}{3}\int\frac{1}{x+1} - \frac{1}{3}\underbrace{\int\frac{4x-9}{x^2+x+3}}_{=:I}$$

$$\int(x-1) + \frac{1}{3}\int\frac{1}{x+1} = \frac{x^2}{2} - x + \frac{1}{3}\ln|x+1|$$

$$I = -\frac{1}{3}\int \frac{4x-9}{x^2+x+3} = -\frac{2}{3}\int\frac{2x+1}{x^2+x+3} + \frac{11}{3}\int\frac{1}{x^2+x+3}$$

$$= -\frac{2}{3}\ln(x^2+x+3) + \frac{11}{3}\underbrace{\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{11}{4}}}_{\frac{2}{\sqrt{11}}\operatorname{arctg}\frac{\left(x+\frac{1}{2}\right)\cdot 2}{\sqrt{11}}}$$

Celkem:

$$\int \frac{x^4+x^3+x^2+x+1}{(x+1)(x^2+x+3)} = \frac{x^2}{2} - x + \frac{1}{3}\ln|x+1| - \frac{2}{3}\ln(x^2+x+3)$$

$$+ \frac{2\sqrt{11}}{3}\operatorname{arctg}\frac{2x+1}{\sqrt{11}} \quad ; \ x \neq -1$$

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(3) $y'' - 2y' + 4y = x^2 e^x$

$$\lambda^2 - 2\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{2 \pm \sqrt{-12}}{2} = 1 \pm i\sqrt{3}$$

$$y_H = c_1 e^x \cos\sqrt{3}x + c_2 e^x \sin\sqrt{3}x$$

$$y_p = (ax^2+bx+c)e^x$$

$$y_p' = (2ax+b+ax^2+bx+c)e^x$$

$$y_p'' = (2a+2ax+b+2ax+b+ax^2+bx+c)e^x$$

$$y_p'' - 2y_p' + 4y_p = (3ax^2+3bx+3c+2a)e^x = x^2e^x$$

$$3a = 1 \qquad a = \frac{1}{3}$$

$$3b = 0 \qquad b = 0$$

$$3c + 2a = 0 \qquad c = -\frac{2}{9}$$

$$y_p = \frac{1}{9}(3x^2 - 2)e^x$$

$$y(x) = c_1 e^x \cos\sqrt{3}x + c_2 e^x \sin\sqrt{3}x + \frac{1}{9}(3x^2-2)e^x$$

$$y(0) = c_1 + 0 - \frac{2}{9} = -\frac{2}{9}$$

$$c_1 = 0$$

Celkem

$$\boxed{y = c_2 e^x \sin\sqrt{3}x + \frac{1}{9}(3x^2-2)e^x}$$

(4) $f(x) = (5x^2-4)e^{-x^2}$

- f spojitá, $D(f) = \mathbb{R}$
- není lichá, periodická
- je sudá

$$\lim_{x \to \pm\infty} (5x^2-4)e^{-x^2} = 0$$

$$f'(x) = 10x\,e^{-x^2} - 2x(5x^2-4)e^{-x^2} = (-10x^3+18x)e^{-x^2}$$
$$= -2x(5x^2-9)e^{-x^2}$$

Sudá $\Rightarrow$ další vyšetřil jen pro $x>0$

$5x^2-9=0 \qquad x = \frac{3}{\sqrt{5}}$

roste na $\left(0, \frac{3}{\sqrt{5}}\right)$

klesá na $\left(\frac{3}{\sqrt{5}}, \infty\right)$

$$f''(x) = \left[(-30x^2+18) - 2x(10x^3+18x)\right]e^{-x^2}$$
$$= (20x^4 - 66x^2 + 18)e^{-x^2}$$
$$= 2\underbrace{(10x^4 - 33x^2 + 9)}\,e^{-x^2}$$

znaménka: $x^2 = \dfrac{33 \pm \sqrt{33^2 - 360}}{20}$

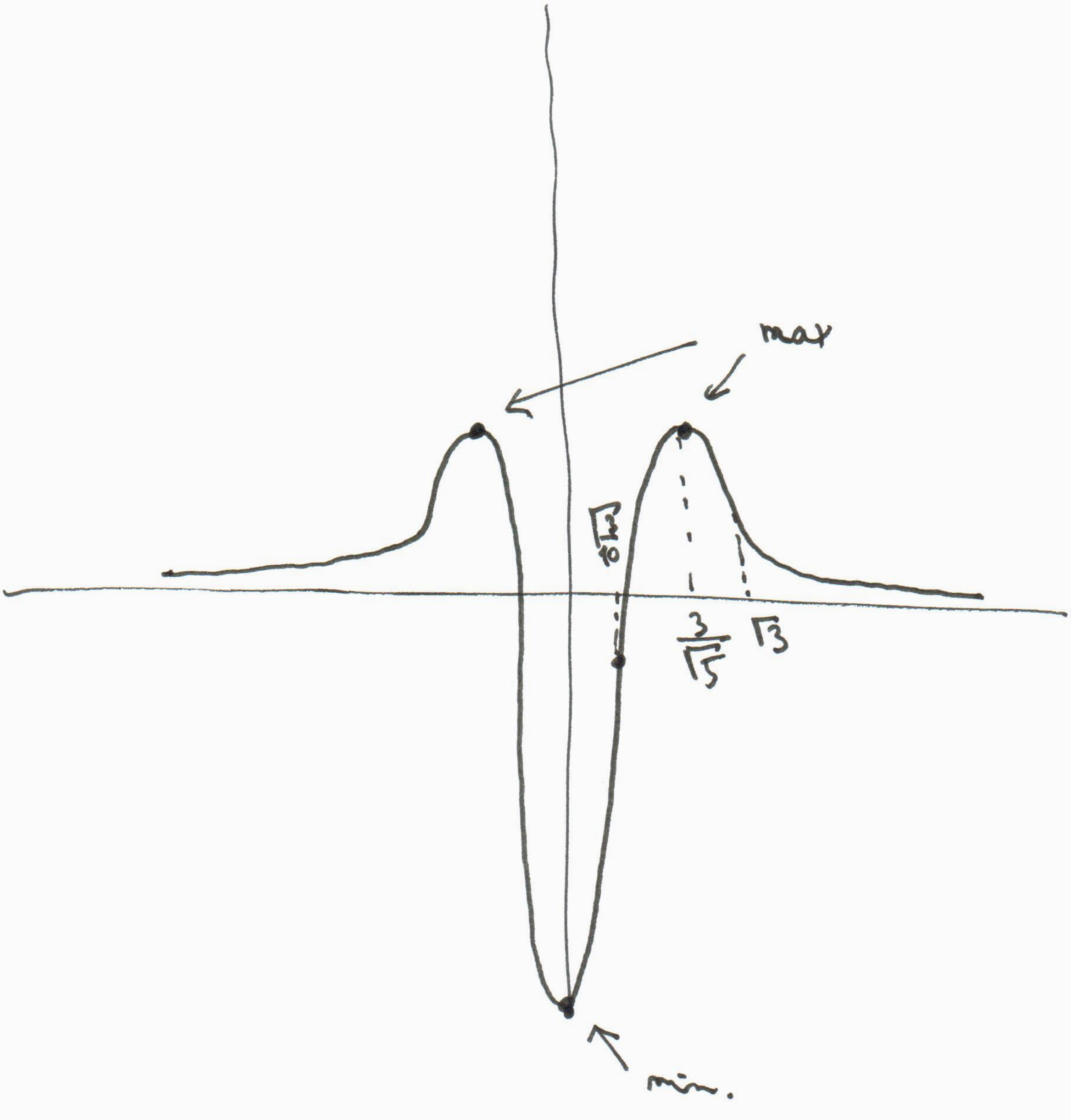
$$= \frac{33 \pm 27}{20} \to \boxed{3},\ \boxed{\tfrac{3}{10}}$$

$x = \pm\sqrt{3}, \pm\sqrt{\frac{3}{10}}$

konvexní na $\left(0, \sqrt{\frac{3}{10}}\right)$

konkávní na $\left(\sqrt{\frac{3}{10}}, \sqrt{3}\right)$

konvexní na $\left(\sqrt{3}, \infty\right)$


max
$\sqrt{\frac{3}{10}}$
$\frac{3}{\sqrt{5}}$
$\sqrt{3}$
min.