

P 13, 25.5.2021

$$x y'' + (\rho + 1 - x) y' - \lambda y = 0 \quad (\text{GRR})$$

$$\rho, \lambda \in \mathbb{C}, \quad \rho \neq -1, -2, -3, \dots$$

\Leftrightarrow

$$\underbrace{\left(-x^{\rho+1} e^{-x} y' \right)'}_{\substack{T y \\ (-p y')'}} = \underbrace{-\lambda x^{\rho} e^{-x} y}_{\lambda p y}$$

$p = x^{\rho} e^{-x}$
 $L^2_{x^{\rho} e^{-x}}(0, \infty)$

$$c_0 = 1$$

$$c_{m+1} = c_m \frac{m + \lambda}{(m+1)(\rho + m + 1)} \quad m = 0, 1, 2, 3, \dots$$

$$\rho \neq -1, -2, -3, \dots$$

$$y \approx \sum_0^{\infty} c_m x^m \quad R = +\infty$$

• INTERMEZZO O HYPERGEOMETRICKÁ ŘADÁČIL

Hg-řada je řada $y = \sum_{n=0}^{\infty} c_n x^n$,

kde:

a) $\exists P, Q$ polynomy s koef. u nej-
vyšší mocniny rovnými 1

$$\text{st } P = p \geq 0, \quad \text{st } Q = q \geq 0$$

Q nemá koef. mezi N a 0 's

$$b) \frac{c_{n+1}}{c_n} = \frac{P(n)}{Q(n)} \cdot \frac{1}{n+1}, \quad n = 0, 1, 2, \dots$$

$$c_0 = 1$$

$$\textcircled{P=Q} \cdot P, Q \equiv 1 \quad \Rightarrow \quad c_{n+1} = \frac{1}{n+1} c_n$$



$$c_0 = 1, c_1 = 1, c_2 = \frac{1}{2!}, c_3 = \frac{1}{3!},$$

$$y = e^x$$

• Řešení GRR (yše) je ve tvaru Hg-řady.

$$c_0 = 1$$

$$\frac{c_{n+1}}{c_n} = \frac{n+1}{(n+1)} \cdot \frac{1}{n+1}$$

pmo - li schepi norlabl P, Q ma koemae' c'nielle, lah:

$$P(n) = (a_1+n)(a_2+n) \dots (a_p+n)$$

$$P(x) = (x-d_1)(x-d_2) \dots$$

$$Q(n) = (b_1+n)(b_2+n) \dots (b_q+n)$$

⇔

$$\frac{c_{n+1}}{c_n} = \frac{(a_1+n) \dots (a_p+n)}{(b_1+n) \dots (b_q+n)} \cdot \frac{1}{n+1}$$

$$\sum_0^\infty c_n x^n = {}_pF_q [a_1, \dots, a_p; b_1, \dots, b_q] (x)$$

$$\begin{bmatrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{bmatrix}$$

Plah':

(i) $p < q+1 \Rightarrow \left| \frac{c_{n+1}}{c_n} \right| \rightarrow 0 \Rightarrow \underline{R = +\infty}$

(ii) $p = q+1 \Rightarrow \left| \frac{c_{n+1}}{c_n} \right| \rightarrow 1 \Rightarrow \underline{R = 1}$

(iii) $p > q+1 \Rightarrow \left| \frac{c_{n+1}}{c_n} \right| \rightarrow \infty \Rightarrow \underline{R = 0}$

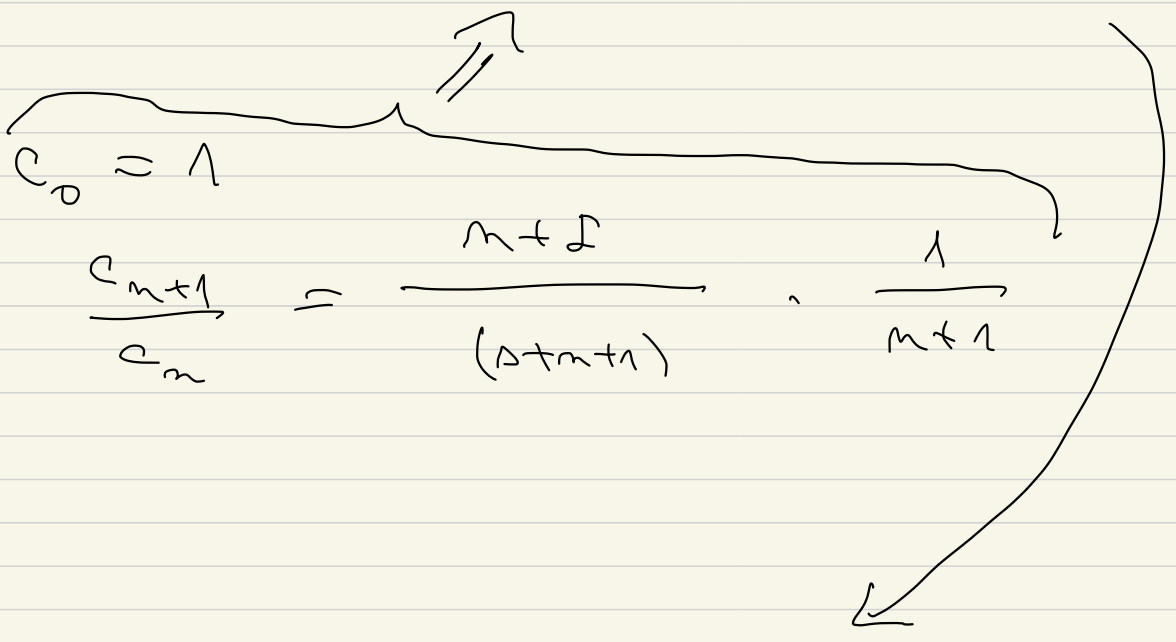
(P_i) GRR: $p = q = 1 \Rightarrow p < q+1 \Rightarrow R = +\infty$

$$y = \text{neser' GRR} = {}_1F_1 [\alpha; \alpha+1] (x)$$

$${}_pF_q [a_1, \dots, a_p; b_1, \dots, b_q](x) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_n}{\prod_{k=1}^q (b_k)_n} \cdot \frac{x^n}{n!}$$

Revised GRD:

$$y = {}_1F_1 [\alpha; \lambda+1](x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\lambda+1)_n} \cdot \frac{x^n}{n!} \quad (*)$$



To find sol. for
 $Ty = \lambda py$

Key from the y above (*) polynomial?

\Rightarrow find $\exists n$, i.e. $(\alpha)_n = 0 \quad \forall k \geq n$

$$(\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1)$$

Polynom $\Leftrightarrow \lambda = -m$ pro nějaké $m \in \mathbb{N}$

Pro vlastí čísla $\lambda = 1, 2, 3, 4, \dots$ ($\lambda = -\alpha$)

leg pro $\alpha = -\lambda = -1, -2, -3, \dots$

jsou odpovídající vlastí kvazipolynomy (*)

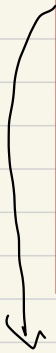
Taylor polynomy se jmenují

Laguerrov polynomy

Def: Laguerrovým polynomem řádu n a stupně α nazýváme polynom definovaný pro $\alpha \in \mathbb{R}, \alpha > -1$ takto:

$$L_n^\alpha(x) := \frac{(\alpha+1)_n}{n!} {}_1F_1[-n, \alpha+1](x)$$

$$= \frac{(\alpha+1)_n}{n!} \sum_{k=0}^n \frac{(-n)_k}{(\alpha+1)_k} \cdot \frac{x^k}{k!}$$



OG reálné polynomy na $L^2_{x^\alpha e^{-x}}(0, \infty)$

Pozn: jde o úplň OG systém, tj jde o bázi.
(Čížák, MA pro FV)

OG dynamické křivky vyjádřené:

1) tzv. explicitní vyjádření

$$L_n^p(x) = \frac{1}{n!} x^{-p} e^x \left(x^{p+n} e^{-x} \right)^{(n)} \quad \checkmark$$

$$\Rightarrow L_0^p(x) = 1$$

$$L_1^p(x) = -x + (p+1)$$

2) rekurentní vzorec

$$x L_n^p(x) = (p+2n+1) L_n^p - (n+1) L_{n+1}^p - (p+n) L_{n-1}^p$$

3) normy

víme:

$$\| \varphi_{n+1} \|^2 = \frac{B_{n+1}}{A_n} \| \varphi_n \|^2$$

$$A_n = n(n+1) \\ B_n = -(p+n)$$

$$A_n \dots \varphi_{n+1}, \quad B_n \dots \varphi_{n-1}$$

$$\Rightarrow \|L_{m+1}^\rho\|_{2,\rho}^2 = \frac{\rho+m+1}{m+1} \|L_m^\rho\|_{2,\rho}^2$$

$$\|L_0^\rho\|_{2,\rho}^2 = \int_0^\infty 1 \cdot x^\rho e^{-x} dx = \Gamma(\rho+1)$$

D. cv.:

$$\|L_m^\rho\|_{2,\rho}^2 = \frac{1}{m!} \Gamma(\rho+m+1) \quad \text{for } m=0,1,2 \dots$$

④ Tzv. vyhovující fce

Def: Vyhovující funkce pro systém fce

$$\varphi_m = \varphi_m(x), \quad m=0,1,\dots$$

navon fci

$$F(x,t)$$

která je analytická v okolí $t=0$,

která splňuje

$$F(x,t) = \sum_{n=0}^{\infty} \varphi_n(x) t^n$$

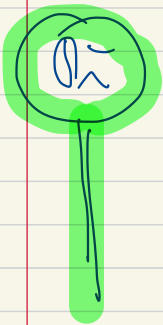
Y kerie f. řad :

$$f \in L^2_{x^\rho e^{-x}}(0, \infty) \Rightarrow \exists c_n \in \mathbb{C}$$

vzhledem $\{L_n^\rho(x)\}$

$$c_n = \frac{(f, L_n^\rho)_{2, \rho}}{\|L_n^\rho\|_{2, \rho}^2}$$

$$f = \sum_{n=0}^{\infty} c_n L_n^\rho$$



Rozvíte pro volné $a \in \mathbb{R}$ fci

e^{-ax} do systému Laguerre-pol. na $(0, \infty)$

Důležitá věta : Každé $f \in L^2_{x^\rho e^{-x}}(0, \infty)$

$$\int_0^\infty (e^{-ax})^2 x^\rho e^{-x} < \infty$$

$$\int_0^\infty x^\rho e^{-(2a+1)x} dx \begin{cases} \text{"0"} & \rho > -1 \checkmark \\ \text{"}\infty\text{"} & : 2a+1 > 0 \checkmark \end{cases}$$

$$e^{-ax} \in L^2_{x^\rho e^{-x}(0, \infty)} \Leftrightarrow a > -\frac{1}{2}$$

Oronly krol:

$$c_m = \frac{1}{\|L_m^\rho\|^2} \int_0^\infty e^{-ax} \cdot \underbrace{L_m^\rho(x)}_{\text{(leysl - vyjãdt)}} \cdot x^\rho e^{-x} dx$$

$$\approx \frac{1}{\cancel{\frac{1}{m!}} \Gamma(\rho+m+1)} \int_0^\infty \cancel{e^{-ax}} \cdot \cancel{x^\rho} \cdot \cancel{e^{-x}} \cdot \left(\cancel{\frac{1}{m!}} \cancel{x^\rho} \cancel{e^{-x}} \left(x^{\rho+m} e^{-x} \right)^{(m)} \right)$$

$$= \frac{1}{\Gamma(\rho+m+1)} \int_0^\infty e^{-ax} \left(x^{\rho+m} e^{-x} \right)^{(m)}$$

m-knal per partes

$$\approx \frac{a^m}{\Gamma(\rho+m+1)} \int_0^\infty e^{-ax} \cdot x^{\rho+m} e^{-x} dx$$

$$(a+1)x = y$$

$$(a+1)dx = dy$$

$$(a > -\frac{1}{2})$$

$$= \frac{a^n}{\Gamma(s+n+1)} \int_0^{\infty} e^{-y} \left(\frac{y}{a+1} \right)^{s+n} \frac{dy}{a+1}$$

$$= \frac{a^n}{\Gamma(s+n+1)} \underbrace{\int_0^{\infty} y^{s+n} e^{-x} \cdot \text{const}}_{\Gamma(s+n+1)} = \frac{1}{(a+1)^{s+n+1}}$$

$$= \frac{1}{(a+1)^{s+1}} \cdot \left(\frac{a}{a+1} \right)^n = c_n$$


$$\Rightarrow a > -\frac{1}{2} \quad x \in (0, \infty)$$

$$e^{-ax} = \frac{1}{(a+1)^{s+1}} \sum_{n=0}^{\infty} \left(\frac{a}{a+1} \right)^n L_n^s(x)$$

$$\textcircled{0} \quad s > 0, a = 1$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{L_n^0(x)}{2^{n+1}}$$



\int  $\rightarrow t = \frac{a}{a+1}$

\Downarrow D.C.V.

$$\frac{1}{(1-t)^{p-1}} e^{-\frac{tx}{1-t}} = \sum_{n=0}^{\infty} L_n^p(x) t^n$$

\equiv
F(x,t)

HOWGH