IV.1 : Jordan-Hölder-Schreier theorem

Definition. A normal series of a group G is a sequence

$$G = G_0 \geq G_1 \geq \dots \geq G_n = 1 \qquad (*)$$

of subgroups of a group G such that $G_{i+1} \trianglelefteq G_i$ for all $i = 0, 1, \dots, n-1$.

- Factors of the normal series $(*)$ are ~~subgroups~~ groups G_i / G_{i+1}, $i = 0, 1, \dots, n-1$.
- The length of the normal series $(*)$ is n (the number of factors).

Hans Julius Zassenhaus
1912 – 1991

Definition: A normal series

$$G = H_0 \geq H_1 \geq \dots \geq H_m = 1$$

is a refinement of a normal series

$$G = G_0 \geq G_1 \geq \dots \geq G_n = 1$$

if $G_0, \dots, G_n$ is a subsequence of $H_0, \dots, H_m$.

Definition: A composition series is a normal series

$$G = G_0 \geq G_1 \geq \dots \geq G_n = 1$$

such that either
- $G_{i+1} = G_i$

or
- G_{i+1} is a maximal normal subgroup of G_i

for all $i = 0, 1, \dots, n-1$.

Definition: Two normal series are equivalent if they have the same length and there is a bijection between their factors such that the corresponding factors are ~~equivalent~~ isomorphic.

Example: $\mathbb{Z}_6 \geq \mathbb{Z}_3 \geq 1$ and $\mathbb{Z}_6 \geq \mathbb{Z}_2 \geq 1$ are two composition series of the cyclic group $\mathbb{Z}_6$. ~~The~~ A bijection between their factors is given by

$$\mathbb{Z}_6 / \mathbb{Z}_3 \overset{\simeq}{\longmapsto} \mathbb{Z}_2 / 1 = \mathbb{Z}_2$$

$$\mathbb{Z}_3 \simeq \mathbb{Z}_3 / 1 \overset{\simeq}{\longmapsto} \mathbb{Z}_6 / \mathbb{Z}_2 .$$

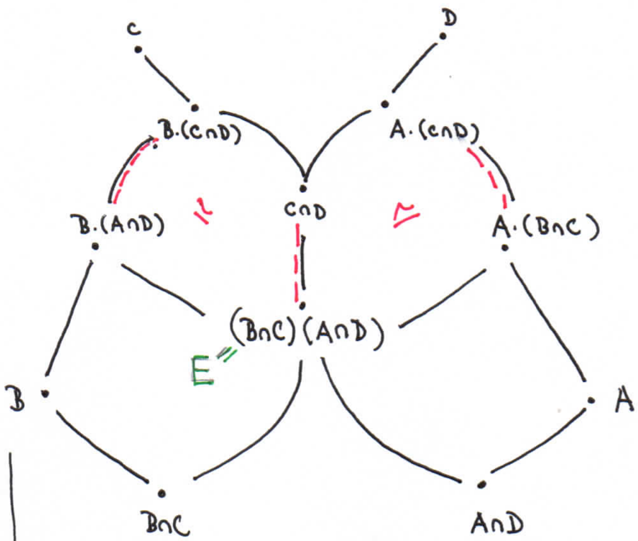
Therefore these two composition series are equivalent.

Lemma (Zassenhaus 1934) 4.1: Let A, B, C, D be subgroups of a group G. If $A \trianglelefteq C$ and $B \trianglelefteq D$, then

$$\bullet \begin{cases} A(C\cap B) \trianglelefteq A\cdot(C\cap D) \\ B\cdot(A\cap D) \trianglelefteq B\cdot(C\cap D) \end{cases}$$

and

$$A(C\cap D)\big/A(C\cap B) \simeq B\cdot(C\cap D)\big/B\cdot(A\cap D)$$



Remark: Zassenhaus lemma is also called Butterfly lemma. The drawing (hopefully) suggest why.

Proof.

Since $A \trianglelefteq C$, $A\cap D \trianglelefteq C\cap D$.

Similarly we get that $B\cap C \trianglelefteq C\cap D$.

From this we get that

$$(B\cap C)(A\cap D) \trianglelefteq C\cap D.$$

Put $E = (B\cap C)(A\cap D)$.

- We prove that $B.(C\cap D)/B.(A\cap D) \simeq C\cap D/E$:

  An element $x \in B\cdot(C\cap D)$ is of the form $b.c$ where $b\in B$ and $c \in C\cap D$. Moreover if $bc = b'c'$ for some $b'\in B$ and $c'\in C\cap D$, then $c'c^{-1} \in B\cap C\cap D \subseteq B\cap C \subseteq E$. Therefore we can define a map
  $$f: B\cdot(C\cap D) \to C\cap D/E$$
  $$b.c \mapsto c.E$$

  - The map f is a homomorphism: Let $b_1, b_2 \in B$, $c_1, c_2 \in C\cap D$. Since B is a normal subgroup of D, $c_1 b_2 = b_2' c_1$ for some $b_2' \in B$. Then $f(b_1c_1b_2c_2) = f(b_1b_2'c_1c_2) = c_1c_2 = f(b_1c_1)f(b_2c_2)$.
  - The map f is onto $C\cap D/E$: Clearly, elements of $C\cap D/E$ are exactly of the form $c.E$, $c\in C\cap D$.
  - $\operatorname{Ker} f = B\cdot(A\cap D)$: $f(bc) = E \Leftrightarrow c\in E \Leftrightarrow c = b'c'$ for some $b'\in B\cap C$, $c'\in A\cap D \Leftrightarrow$
  $$bc = \underbrace{bb'}_{\in B}c' \in B\cdot(A\cap D).$$

  Therefore f factors through an isomorphism $\varphi: B(C\cap D)/B(A\cap D) \to C\cap D/E$.

- Similarly we prove that $A(C\cap D)/A(C\cap B) \simeq C\cap D/E$.

We conclude that

$$A(C\cap D)\big/A(B\cap C) \simeq C\cap D/E \simeq B(C\cap D)\big/B(A\cap D).$$

$\square$

Theorem 4.2 (Schreier 1928): Every two normal series of a group G have equivalent refinements.

Otto Schreier
1901 – 1929

Proof: Let

$$G = G_0 \geq G_1 \geq \ldots \geq G_n = 1 \quad \text{and} \quad G = H_0 \geq H_1 \geq \ldots \geq H_m = 1$$

be two composition series of a group G (in particular, $G_{i+1} \trianglelefteq G_i$ for all $i = 1, \ldots, n-1$ and $H_{j+1} \trianglelefteq H_j$ for all $j = 1, \ldots, m-1$).

For every $i \leq n$, $j \leq m$ put

- $G_{i,j} = G_{i+1}(G_i \cap H_j)$ and
- $H_{i,j} = H_{j+1}(G_i \cap H_j)$.

Observe that:

- for every $i = 0, \ldots, n-1$ we have a sequence

$$G_i = G_{i,0} = G_{i+1}(\underbrace{G_i \cap H_0}_{= G_i}) \geq G_{i,1} = G_{i+1}(G_i \cap H_1) \geq \ldots \geq G_{i,j} = G_{i+1}(G_i \cap H_j) \geq \ldots \geq G_{i,m} = G_{i+1}(\underbrace{G_i \cap H_m}_{= 1}) = G_{i+1}$$

- for every $j = 0, \ldots, m-1$ we have a sequence

$$H_j = H_{0j} = H_{j+1}(\underbrace{G_0 \cap H_j}_{= H_j}) \geq H_{1j} = H_{j+1}(G_1 \cap H_j) \geq \ldots \geq H_{ij} = H_{j+1}(G_i \cap H_j) \geq \ldots \geq H_{nj} = H_{j+1}(\underbrace{G_n \cap H_j}_{= 1}) = H_{j+1}$$

Applying Zassenhaus lemma for $A = G_{i+1}$, $B = H_{j+1}$, $C = G_i$, $D = H_j$, we get that

$$\underbrace{G_{i+1} \cdot (G_i \cap H_{j+1})}_{= A(C \cap B)} = G_{i,j+1} \trianglelefteq G_{i,j} = \underbrace{G_{i+1}(G_i \cap H_j)}_{= A(C \cap D)}$$

Therefore, we have a composition series:

$$G = G_0 = G_{0,0} \geq G_{0,1} \geq \ldots \geq G_{0,m} = G_{1,0} \geq G_{1,1} \geq \ldots \geq G_{1,m} = G_{2,0} \geq \ldots \geq G_{n-2,m} = G_{n-1,0} \geq G_{n-1,1} \geq \ldots \geq G_{n-1,m} = 1 \qquad (*)$$

Applying Zassenhaus lemma again for $A = G_{i+1}$, $B = H_{j+1}$, $C = G_i$, $D = H_j$, we get that

$$\underbrace{H_{j+1} \cdot (H_j \cap G_{i+1})}_{= B \cdot (D \cap A)} = H_{i+1,j} \trianglelefteq H_{i,j} = \underbrace{H_{j+1}(H_j \cap G_i)}_{= B \cdot (D \cap B)}$$

and we have a composition series

$$G = H_0 = H_{0,0} \geq H_{1,0} \geq \ldots \geq H_{n,0} = H_{0,1} \geq H_{1,1} \geq \ldots \geq H_{n,1} = H_{0,2} \geq \ldots \geq H_{n,m-2} = H_{0,m-1} \geq H_{1,m-1} \geq \ldots \geq H_{n,m-1} = 1 \qquad (**)$$

The isomorphism of quocients in Zassenhaus lemma gives us that

$$\frac{B(D\cap C)}{B(D\cap A)} \cong \frac{A(C\cap D)}{A(B\cap D)}$$

$$H_{i,j}/H_{i+1,j} = \frac{H_{j+1}(H_j\cap G_i)}{H_{j+1}(H_j\cap G_{i+1})} \cong G_{i+1}(G_i\cap H_j)/G_{i+1}(G_i\cap H_{j+1}) = G_{ij}/G_{i,j+1}$$

for all $i=0,1,\dots,n-1$, $j=0,\dots,m-1$.

So the corresponding factors in both series (*) and (**) are isomorphic. Therefore the refinements (*) of $G=G_0\geq\dots\geq G_n=1$ and (**) of $H=H_0\geq\dots\geq H_m=1$ are equivalent. $\square$

Theorem 4.3: (Jordan - Hölder) Every two composition series of a group G are equivalent.

Proof. Composition series are normal series that are maximal strictly decreasing.

Suppose that $G=G_0>G_1>\dots>G_n=1$ is a composition series. Any refinement of the series has exactly n non-trivial (= not isomorphic to the one-element group) factors. They correspond to the factors $G_0/G_1, G_1/G_2,\dots, G_{n-1}/G_n$.

If $G=H_0>H_1>\dots>H_m=1$ is another composition series of the group G, then the two series have equivalent refinements (by Schreier's theorem). Therefore $m=n$ and there is a bijection between their factors such that the corresponding factors are isomorphic (since the remaining factors in the equivalent refinements are trivial). It follows that the composition series are equivalent. $\square$

IV.2. Solvable groups.

Definition. • A <u>solvable series</u> of a group G is a normal series with abelian factors.

• A group is <u>solvable</u> if it has a solvable series.

Theorem 4.4: 1. A subgroup of a solvable group is solvable.

2. A quotient of a solvable group is solvable.

Proof: 1. Let H be a subgroup of a group G. Suppose that the group G has a solvable series

$$G = G_0 \geq G_1 \geq \dots \geq G_n = 1 .$$

> Recall: The Second isomorphism theorem:
> Let $H \trianglelefteq G$ and $B \leq G$. Then $B \cap H \trianglelefteq B$ and $B/B\cap H \simeq BH/H$

Consider the series

$$H = \overset{H\cap G_0}{H_0} \geq H \cap G_1 \geq \dots \geq H \cap G_n = 1$$

For every $i = 0, \dots, n-1$ we have that

$\underset{H'}{G_{i+1}} \trianglelefteq G_i$ and $\underset{B}{H \cap G_i} \trianglelefteq G_i$

Applying the Second isomorphism theorem, we get that $\underbrace{\overbrace{(H\cap G_i)}^{B} \cap \overbrace{G_{i+1}}^{H'}}_{H\cap G_{i+1}} \trianglelefteq \overbrace{H \cap G_i}^{B}$ and

$$\overbrace{H\cap G_i}^{B} \Big/ \underbrace{H \cap G_{i+1}}_{B\cap H'} \simeq \overbrace{G_{i+1}(H \cap G_i)}^{B\,\cdot\, H'} \Big/ \underbrace{G_{i+1}}_{H'} \leq G_i/G_{i+1} .$$

Since the group G_i/G_{i+1} is abelian, then $H\cap G_i/H \cap G_{i+1}$, isomorphic to its subgroup is abelian as well.

2) Let G be a solvable group with a ~~ser~~ solvable series

$$G = G_0 \geq G_1 \geq \dots \geq G_n = 1 .$$

Let $\phi : G \longrightarrow H$ be a homomorfism. Put $K = \ker \phi = \{ g \in G \mid \phi(g) = 1 \}$.

Fix $i \in \{0, \dots, n-1\}$.

Since $K \trianglelefteq G$, $gK = Kg$ for all $g \in G$. In particular $G_{i+1}K = KG_{i+1}$. Since $G_{i+1} \trianglelefteq G_i$, we have for every $g' \in G_i$ that

$$g' K (G_{i+1}K) K g'^{-1} = g' G_{i+1} g'^{-1} \cdot K = G_{i+1}K$$

> Recall: The ~~first~~ third isomorphism theorem:
> Let $A \leq B \leq G$, both A, B be normal in G. Then $B/A \trianglelefteq G/A$ and
> $$G/A \Big/ B/A \simeq G/B .$$

Therefore $G_{i+1}K \trianglelefteq G_i K$.

We have that $K \trianglelefteq G_{i+1}K \trianglelefteq G_iK$, $K \trianglelefteq G_iK$ and by the third isomorphism theorem

$$G_{i+1}\cdot K/K \trianglelefteq G_i \cdot K/K \text{ and}$$

By the second isomorphism theorem (using $G_iK = G_i(G_{i+1}K)$)

$$G_i\cdot K/K \Big/ G_{i+1}K/K \simeq G_iK/G_{i+1}K \simeq G_i/G_i \cap G_{i+1}K$$

$G_i/G_i \cap G_{i+1}K$ is isomorphic to a factor of the abelian group G_i/G_{i+1}. Therefore it is abelian. Now observe that $G_iK/K \simeq \phi(G_i)$ and $G_{i+1}K/K \simeq \phi(G_{i+1})$. Therefore $\phi(G_{i+1}) \trianglelefteq \phi(G_i)$ and $\phi(G_i)/\phi(G_{i+1}) \simeq G_i/G_i \cap$ ~~$G_{i+1}K$~~ $G_{i+1}K$ which is abelian.

We conclude that

$$H = \phi(G_0) \geq \phi(G_1) \geq \dots \geq \phi(G_n) = 1 \qquad \text{is a normal series of } H.$$

$\square$

Theorem 4.5: If $H \trianglelefteq G$ and both H and G/H are solvable, then G is solvable.

Proof: Let

$$G/H = J_0 \geq J_1 \geq \dots \geq J_m = 1$$

be a solvable series for the group G/H and

$$H = H_0 \geq H_1 \geq \dots \geq H_n = 1$$

be a solvable series for the group H.

Let $\phi : G \to G/H$ be a canonical homomorphism. Put $G_i = \phi^{-1}(J_i)$ for all $i = 0, \dots, m$. In particular, $G_m = H_0 = H$. Let $i < m$. For every $g \in G_i$,

$$\phi(g \cdot G_{i+1} g^{-1}) = \phi(g) \cdot \phi(G_{i+1}) \phi(g^{-1}) = \phi(g) \cdot J_{i+1} \phi(g)^{-1} = J_{i+1} = \phi(G_{i+1})$$

since $\phi(g) \in J_i$ and $J_{i+1} \trianglelefteq J_i$. Therefore $g G_{i+1} g^{-1} \subseteq G_{i+1}$. We conclude that $G_{i+1} \trianglelefteq G_i$. Moreover $G_i / G_{i+1} \cong \frac{G_i/H}{G_{i+1}/H} \cong J_i / J_{i+1}$, which is an abelian group. We conclude that

$$G = G_0 \geq G_1 \geq \dots \geq G_m = H_0 \geq H_1 \geq \dots \geq H_n = 1$$

is a ~~normal~~ solvable series for G. □

Corollary 4.6: Let $H \trianglelefteq G$. Then the group G is solvable iff both H and G/H is solvable.

Theorem 4.7: Every finite p-group is solvable.

Proof. By induction on the size of the p-group G. Since G is a p-group, $Z(G) \neq 1$. Then $G/Z(G)$ is a smaller p-group, ~~the~~ hence solvable by the induction hypothesis. ~~Let $G/Z(G) = G_0 \geq \dots \geq G_n = Z(G)\ 1$~~ Since $Z(G)$ is abelian, hence solvable, G is solvable. □

Definition:

- Let a, b are elements of a group G. A commutator of a, b is the element

$$[a,b] = a b a^{-1} b^{-1}$$

- The commutator subgroup of the group G is the subgroup

$$G' = [G, G]$$

generovaná všemi $[a,b]$, $a, b \in G$.

Theorem 4.8: ① The commutator subgroup $G' = [G,G]$ is a normal subgroup of G.

② Let H be a normal subgroup of G. The quotient G/H is abelian iff $G' \leq H$.

Proof: ① $G' \trianglelefteq G$ follows from the equality $x[g,h]x^{-1} = [xgx^{-1}, xhx^{-1}]$, for all $g, h, x \in G$.

② For $g \in G$ let $\bar{g}$ denote the coset gH.

($\Leftarrow$) Suppose that $G' \leq H$. Then $[g,h] = ghg^{-1}h^{-1} \in H$ for all $g, h \in G$. It follows that $\bar{1} = \overline{ghg^{-1}h^{-1}} = \bar{g}\bar{h}\bar{g}^{-1}\bar{h}^{-1}$, hence $\bar{g}\bar{h} = \bar{h}\bar{g}$ in G/H. Therefore the quotient G/H is abelian. ($\Rightarrow$) If G/H is abelian, the $\bar{g}\bar{h}\bar{g}^{-1}\bar{h}^{-1} = \overline{ghg^{-1}h^{-1}} = \bar{1}$ for all $g, h \in G$. From this we infer that $G' \leq H$. $\square$

We can strenghten the previous theorem ① as follows:

Lemma 4.9: If $H \trianglelefteq G$, then $H' \trianglelefteq G$.

Proof: For all $h, h' \in H$ and all $x \in G$: $x[h,h']x^{-1} = [xhx^{-1}, xh'x^{-1}]$. It follows that the set of generators $\{[h,h'] \mid h, h' \in H\}$ of H' is invariant w.r.t. conjugation by elements from G, hence $H' \trianglelefteq G$. $\square$

Definition: We define inductively

- $G^{(0)} = G$,
- $G^{(i+1)} = [G^{(i)}, G^{(i)}]$,

for all $i = 0, 1, 2, \dots$

- The groups $G^{(i)}$, $i = 0, 1, 2, \dots$ are called <u>higher commutators</u> of G.
- The sequence $G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \dots \geq G^{(i)} \geq \dots$ is called the <u>derived sequence</u> of G.

It follows from the previous lemma:

Lemma 4.10: The higher commutators are normal subgroup of the group G.

Lemma 4.12: If $G = G_0 \geq G_1 \geq G_2 \geq \ldots \geq G_n = 1$ is a solvable series, then $G_i \geq G^{(i)}$ for all i.

Proof: We proceed by induction. Suppose that $G_i \geq G^{(i)}$. Then $G_i' = [G_i, G_i] \geq [G^{(i)}, G^{(i)}] = G^{(i+1)}$. Since the quotient G_i / G_{i+1} is abelian, $G_{i+1} \geq G_i'$. Because $G_0 = G \geq G = G^{(0)}$, we are done. $\square$

Lemma 4.13: A group G is solvable iff $G^{(n)} = 1$ for some n.

Proof: ($\Leftarrow$) If $G^{(n)} = 1$, then $G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \ldots \geq G^{(n)} = 1$ is a solvable series for G.

($\Rightarrow$) If $G = G_0 \geq G_1 \geq G_2 \geq \ldots \geq G_n = 1$ is a solvable series for G, then $1 = G_n \geq G^{(n)}$. $\square$

Let us finish with some "deep" theorems without a proof:

Definition: Let p be a prime. If G is a finite group of order mp^n, where $p \nmid m$, then a <u>p-complement</u> of G is a subgroup of G of order m.

Theorem 4.15 (P. Hall, 1937): If G is a finite group having a p-complement for every p, then G is solvable.

Corollary 4.16 (Burnside): A group of an order $p^m q^n$, where p, q are primes, is solvable.

Theorem 4.17 (Feit, Thompson): A finite group of odd order is solvable.

Theorem 4.18 (P. Hall, 1928): Let G be a solvable group of order mn. If m and n are relatively prime, then G contains a subgroup of order m and all subgroups of order m are conjugated.