V. (Nilpotentní grupy) NILPOTENT GROUPS

Definition. Let H, K be subgroups of a group G. We put

$$[H,K] = \langle \{ [h,k] = hkh^{-1}k^{-1} \mid h \in H \,\&\, k \in K \} \rangle.$$

Otto Hölder
1859 – 1937

- $G' = [G,G]$
- $G^{(n+1)} = [G^{(n)}, G^{(n)}]$

Definition: Let $H \leq G$. The centralizer of H in G is the subgroup

$$C_G(H) = \{ x \in G \mid [x,h] = 1 \text{ for all } h \in H \}.$$

- The subgroup $K \leq G$ centralizes H if $K \leq C_G(H)$.
- The subgroup $K \leq G$ normalizes H if $K \leq N_G(H)$.

- K centralizes H iff $[K,H] = 1$.
- K normalizes H iff $[K,H] \leq H$.

Lemma 5.1:

1. Let $K \leq H \leq G$ and $K \trianglelefteq G$. Then $[H,G] \leq K$ iff $H/K \leq Z(G/K)$.
2. Let $H, K \leq G$. If $\phi : G \to G_1$ is a homomorphism, then $f([H,K]) = [f(H), f(K)]$.

Proof:

1. Let $h \in H$, $g \in G$. Then

$$hKgK = gKhK \quad \text{iff} \quad \underbrace{(hgh^{-1}g^{-1})}_{[h,g]}K = K \quad \text{iff} \quad [h,g] \in K.$$

2. $f([H,K]) = \langle \{ f([h,k]) \mid h \in H, k \in K \} \rangle$ while $[f(H), f(K)] = \langle [f(h), f(k)] \mid h \in H, k \in K \rangle$.

But $f([h,k]) = f(hkh^{-1}k^{-1}) = f(h)f(k)f(h)^{-1}f(k)^{-1} = [f(h), f(k)]$.

$\square$

- Put:
  - $\gamma_1(G) = G$
  - $\gamma_{i+1}(G) = [\gamma_i(G), G]$

Definition: The lower central series of G is the series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \dots$$

⚠ It does not need to reach 1.

- By induction we verify that $\gamma_i(G) \trianglelefteq G$: for all $g \in \gamma_{i-1}(G)$, $h \in G$, $x \in G$:

$$x[g,h]x^{-1} = [\underbrace{xgx^{-1}}_{\substack{\cap \\ \gamma_{i-1}(G) \text{ by ind. hypothesis}}}, xhx^{-1}] \in [\gamma_{i-1}(G), G]$$

- Put
  - $\xi^0(G) = 1$
  - $\xi^{i+1}(G)$ = the largest subgroup of G s.t. $\xi^{i+1}(G)/\xi^i(G) = Z\left(G/\xi^i(G)\right)$.

Definition: The ~~th~~ upper central series of G is the series

$$1 = \xi^0(G) \leq \underset{\substack{\| \\ Z(G)}}{\xi^1(G)} \leq \xi^2(G) \leq \dots$$

◁ $g \in \xi^{i+1}(G)$ iff $[g,h] \in \xi^i(G)$ for all $h \in G$.

- By induction we prove that $\xi^i(G) \trianglelefteq G$. For all $g, h, x \in G$

$$[g,h] \in \xi^i(G) \;\&\; \xi^i(G) \trianglelefteq G \Rightarrow x[g,h]x^{-1} \in \xi^i(G) \Rightarrow [xgx^{-1}, xhx^{-1}] \in \xi^i(G).$$

But $\{xhx^{-1}, h \in G\} = G$.

Lemma 5.2: Let G be a group. For an integer c: $\xi^c(G) = G$ iff $\gamma_{c+1}(G) = 1$. Moreover, in this case $\gamma_{i+1}(G) \leq \xi^{c-i}(G)$ for all $i \leq c$.

$$\boxed{\begin{array}{ccccccccc} 1 = \xi^0(G) & \leq & \xi^1(G) & \leq & \dots & \leq \xi^{c-1}(G) \leq & \xi^c(G) & = & G \\ \| & & \text{VI} & & & \text{VI} & \| & & \\ 1 = \gamma_{c+1}(G) & \leq & \gamma_c(G) & \leq & \dots & \leq \gamma_2(G) \leq & \gamma_1(G) & = & G \end{array}}$$

If the group G is clear from the context, we will simplify the notation to ξ^i and γ_i.

Proof:

- Assume first that $\zeta^c = G$.

We prove by induction on i that $\gamma_{i+1} \leq \zeta^{c-i}$, for every $i \in \{0,\dots,c\}$

- $i=0$: Then $\gamma_1 = G \leq G = \zeta^c$ (from the definition)
- $i \to i+1$: Suppose that $\gamma_{i+1} \leq \zeta^{c-i}$.

By the definition $\zeta^{c-i}/\zeta^{c-i-1} = Z\left(G/\zeta^{c-i-1}\right)$, hence $[\zeta^{c-i}, G] \leq \zeta^{c-i-1}$.

Applying the induction hypothesis, we get that

$$\gamma_{i+2} = [\gamma_{i+1}, G] \leq [\zeta^{c-i}, G] \leq \zeta^{c-i-1}.$$

For $i=c$ we get that $\gamma_{c+1} \leq \zeta^0 = 1$, hence $\gamma_{c+1} = 1$.

- Now assume that $\gamma_{c+1} = 1$.

We prove by induction on j that $\gamma_{c+1-j} \leq \zeta^j$, for every $j \in \{0,\dots,c\}$.

- $j=0$: then $1 = \gamma_{c+1} \leq \zeta^0 \ (=1)$
- $j \to j+1$: Suppose that $\gamma_{c+1-j} \leq \zeta^j$. There is a surjective homomorphism

$$\phi: G/\gamma_{c+1-j} \to G/\zeta^j, \quad g\cdot\gamma_{c+1-j} \mapsto g\cdot\zeta^j.$$

By the definition $\gamma_{c+1-j} = [\gamma_{c-j}, G]$

Therefore

$$\gamma_{c-j}/\gamma_{c+1-j} = \gamma_{c-j}/[\gamma_{c-j}, G] \leq Z\left(G/[\gamma_{c-j}, G]\right) = Z\left(G/\gamma_{c+1-j}\right)$$

Every surjective group homomorphism maps centrum to a centrum of the target group. Therefore

$$\Phi\left(\gamma_{c-j}/\gamma_{c-j+1}\right) \leq \Phi\left(Z\left(G/\gamma_{c+1-j}\right)\right) \leq Z\left(G/\zeta^j\right) = \zeta^{j+1}/\zeta^j$$

It follows that $\gamma_{c-j}\zeta^j \leq \zeta^{j+1}\zeta^j = \zeta^{j+1}$, hence $\gamma_{c-j} \leq \zeta^{j+1}$.

For $j=c$ we get that $G = \gamma_1 \leq \zeta^c$, hence $\zeta^c = G$.

$\square$

Definition: A group G is nilpotent if there is c such that $\gamma_{c+1}(G) = 1$.

The least c such that $\gamma_{c+1}(G) = 1$ is called the nilpotence class of G.

Lemma 6.3: Every finite p-group is nilpotent.

Proof: Every quocient of a ~~nilpotent group is nilpotent~~ p-group is a p-group and every finite p-group has a non-trivial center. If $\zeta^i(G) < G$ for some i, then $Z\left(G/\zeta^i(G)\right) \neq 1$, hence $\zeta^i(G) < \zeta^{i+1}(G)$. As G is finite, the upper central series terminates at G. $\square$

Lemma 5.4: Every nilpotent group is solvable.

Proof: It follows from $G^{(i)} \leq \gamma_i(G)$ for all i. $\square$

$\triangleleft$ A nilpotent group has a non trivial center.

Example: S_3 is a solvable group that is not nilpotent.

Lemma 5.5: Let H be a subgroup of a group G. If G is nilpotent of a class c, then H is nilpotent of class $\leq c$.

Proof: By induction we prove that $\gamma_i(H) \leq \gamma_i(G)$ for all i. $\square$

Lemma 5.6: If G is nilpotent of class c and $H \trianglelefteq G$, then G/H is nilpotent of class $\leq c$.

Proof: Let $f: G \to G/H$ be a canonical projection. Then $\gamma_i(G/H) \leq f(\gamma_i(G))$. $\square$

$\triangleleft$ If H, K are groups, then $\gamma_i(H \times K) \leq \gamma_i(H) \times \gamma_i(K)$.

Lemma 5.7: If H, K are nilpotent, then $H \times K$ is nilpotent.