Introduction to Group Theory (NMAG337)

Exercise sheet 11

20. 12. 2022

(4 days until Christmas)

Exercise 1. Let G be a finite simple group, H a nontrivial subgroup of G. Show that if

- $|Conj_G(H)| = n$, then $n! \geq |G|$.
- $[G : H] = n$, then $n! \geq |G|$.

Hint: consider the action of G on $Conj_G(H)$ or the cosets of H. You can also show that $[G : H] \geq |Conj_G(H)|$ and solve the second part using the first part.

Exercise 2. Suppose that in the previous exercise $n \neq 2$. (In other words, suppose that A_n is a nontrivial normal subgroup of S_n.) Show that $\frac{n!}{2} \geq |G|$ and that the equality can hold only if $G \cong A_n$. Use this to show that there is no injective homomorphism $S_n \to A_{n+1}$.

Exercise 3. Show that the group G of rotations of a cube is isomorphic to S_4 and the group H of all of its symmetries is isomorphic to $S_4 \times \mathbb{Z}_2$

- Compare the orders of the groups that should be isomorphic and check that they are equal.
- Consider the action of the group of rotations on the set of 4 diagonals of the cube. Show that this homomorphism $G \to S_4$ is injective and conclude that it is an isomorphism.
- Find a normal 2-element subgroup of H, use it to find the isomorphism $H \cong S_4 \times \mathbb{Z}_2$.

Exercise 4. Show that the group G of rotations of a regular dodecahedron (the Platonic solid with 12 faces) is isomorphic to A_5 and the group H of all of its symmetries is isomorphic to S_5:

- Compare the orders of the groups that should be isomorphic and check that they are equal.
- On the union of edges of the dodecahedron, define a metric $d(A, B)$ as the length of the shortest path from A to B going only on the edges. Let X be the set of the 30 centres of edges. Define a relation $\sim$ on X as $A \sim B \iff d(A, B) \in \{0, 3, 6\}$. Show that $\sim$ is an equivalence relation.
- Observe that the group of rotations/symmetries acts on the 5-element set $X/\sim$. Show that these homomorphisms $G \to S_5$, $H \to S_5$ are injective. Conclude the result.

Note: the group of rotations/symmetries is the same for a cube and an octahedron and for a dodecahedron and an icosahedron since you get one platonic solid from the other by connecting the centres of faces.

Exercise 5. Let $\varphi : \mathbb{Z}_2 \to Aut(S_3)$ be defined by $1 \mapsto (\pi \mapsto (1\ 2)\pi(1\ 2))$. Show that $S_3 \rtimes_\varphi \mathbb{Z}_2 \cong S_3 \times \mathbb{Z}_2$. *(Compare this with Ex. 4, Sh. 7: $H \rtimes_\varphi K$ is abelian if and only if H, K are abelian and $\varphi_k = id_H$ for all $k \in K$)*

Exercise 6. It can be shown that a subgroup of a finitely generated abelian subgroup is finitely generated. Show that this doesn't hold for groups in general: let G be the subgroup of $S_\mathbb{Z}$ generated by $a \mapsto a + 1$ and the transposition $(1\ 2)$, find a subgroup of G, which is not finitely generated.