Introduction to Group Theory (NMAG337)

Exercise sheet 2

11. 10. 2022

Exercise 1. Show that a group of order $p_1^{n_1} \cdot \cdots \cdot p_k^{n_k}$ can be generated by $\sum_{i=1}^{k} n_k$ elements.

Exercise 2. Suppose $G = \langle x_1, \ldots, x_n \rangle$ and $x_i x_j = x_j x_i$ for all $1 \leq i, j \leq n$. Show that G is abelian.

Exercise 3. Find the center $Z(G) = \{x \in G; \forall g \in G\ xg = gx\}$ of the group $G = D_{2n}$ (the $2n$ element group of symmetries of a regular n-sided polygon).

Exercise 4. Find the subgroups of D_{12} (the group of symmetries of a hexagon):

- Find the order of each element of the group.
- Use this to determine all the cyclic subgroups.
- Let x, y be two elements, which are not in a common cyclic subgroup. How does the Lagrange's theorem restrict the possible sizes of $\langle x, y \rangle$? For which x and y this guarantees $\langle x, y \rangle = D_{12}$?
- Suppose x is a rotation, y is a reflection and $ord(x) \neq 6$. Find a geometric object in the plane, which is conserved by both x and y (like the hexagon is), but it is not conserved by some other element of D_{12}. Use this object to show that $\langle x, y \rangle \neq G$.
- Suppose that x and y are both reflections. Show that $\langle x, y \rangle$ must contain some non-trivial rotation. From that conclude that $\langle x, y \rangle$ is either G or one of the already found subgroups.
- List all of the subgroups of D_{12}.

Exercise 5. Find the subgroups of A_4:

- Find the order of elements and the cyclic subgroups.
- Show that A_4 has no subgroup H of order 6. *Hint: According to previous exercise sheet, any such subgroup would be normal. Based on the order of $x \in A_4$, what could be its image in the map $G \to G/H$?*
- Find all of the subgroups of A_4.

Exercise 6. Show that a group G of order p^2 is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$:

- If there exists an element of order p^2, show the isomorphism.
- Suppose there isn't such element. What are the possible orders of elements of G?
- How many non-trivial subgroups does G have?
- Let X be the set of all non-trivial subgroups of G. Show that for $g \in G$, $\pi_g : X \to X$, $H \mapsto gHg^{-1}$ is a permutation on X.
- Show that $\pi : g \mapsto \pi_g$ is a homomorphism of groups $G \to S_X$.
- From the orders of G and S_X determine how many elements can $Im(\pi)$ have. Use this to show that G has a non-trivial normal subgroup H.
- Similarly define a conjugation homomorphism $G \to S_{H \setminus \{e\}}$. Use it to show that $H \subseteq Z(G)$.
- Show that G is abelian and isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.