Introduction to Group Theory (NMAG337)

Exercise sheet 3

18. 10. 2022

Exercise 1. Let G be group. Show that $G/Z(G)$ cannot be nontrivial cyclic.

Exercise 2. Using the Burnside's theorem, count how many graphs on five vertices are there (up to isomorphism).

Exercise 3. How many binary relations there exists on a 4-element set (up to isomorphism)?

Exercise 4. How many binary operations there exists on a 3-element set (up to isomorphism)?

Exercise 5. How many colorings (of faces) of a tetrahedron by n colors are there up to rotation? Up to rotation and reflection?

Exercise 6. Let f be a nontrivial rotation in 3D, i.e. a non-neutral element of $SO(3) := \{M \in \mathbb{R}^{3\times 3}; M^T M = I, det(M) = 1\}$. Prove that it has a unique axis of rotation (i.e. an eigenspace corresponding to the eigenvalue 1 has dimension 1).

Exercise 7. Let $n > 4$. Show that A_n is generated by 3-cycles.

Exercise 8. Let $n > 4$, $N \trianglelefteq A_n$. Show that $N = A_4$ if:

- N contains a 3-cycle.
- N contains a permutation, which (in cyclic notation) has a cycle of length > 3.
- N contains a permutation, which has exactly one 3-cycle.
- N contains a permutation, which has at least two 3-cycles.
- N contains a permutation, which has some 2-cycle.

Conclude that A_n is simple for $n \in \mathbb{N} \setminus \{1, 2, 4\}$.

Exercise 9. Let $A = \bigcup A_n$, i.e. a group consisting of bijections $f : \mathbb{N} \to \mathbb{N}$, such that $f(n) \neq n$ only for finitely many elements and this permutation is even. Show that A is simple.