Introduction to Group Theory (NMAG337)

Exercise sheet 6

8. 11. 2022

Definition. Let H, K be groups, $\varphi : K \to Aut(H)$ a group homomorphism. Then the semidirect product denoted by $H \rtimes_\varphi K$ (or sometimes just $H \rtimes K$) is the group with the underlying set $H \times K$ and multiplication defined by $(h, k) \cdot (h', k') = (h \cdot \varphi_k(h'), k \cdot k')$.

Exercise 1. Show that the semidirect product is indeed a group and that the direct product is a special case of the semidirect product.

Exercise 2. Observe that $H \rtimes_\varphi K$ has a subgroup K (or $\{e\} \times K$ to be precise), a normal subgroup H (similarly here), $G = HK$ and $H \cap K = \{e\}$.

Exercise 3. Conversely, let G be a group, $H \trianglelefteq G$, $K \leq G$, $KH = G$ and $K \cap H = \{e\}$. Show that $G \cong H \rtimes_\varphi K$ for some $\varphi : K \to Aut(H)$.

Exercise 4. Show that $H \rtimes_\varphi K/H \cong K$.

Bonus If you know what an "exact sequence" and "split exact sequence" is, observe that

$$0 \to H \to H \rtimes_\varphi K \to K \to 0$$

is a something like a "half-split" exact sequence, in the sense that there exists the one-sided inverse $K \to H \rtimes_\varphi K$, but (usually) not the one-sided inverse $H \rtimes_\varphi K \to H$. In this sense, a direct product is an exact sequence, which splits on both sides, and a triple (normal subgroup, group, factor) is an exact sequence, which doesn't have to split at all.

Exercise 5. Show that $D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$. Use this to find an isomorphism $D_{4n}/\{id, rot_{180^\circ}\} \cong D_{2n}$ (this isomorphism was already in the Sheet 4, but this time you don't need any tricks).

Exercise 6. Show that the quaternion group Q cannot be written as a semidirect product of smaller groups.

Exercise 7. Let $\psi \in Aut(K)$. Show that $H \rtimes_\varphi K \cong H \rtimes_{\psi \circ \varphi} K$.

Exercise 8. Let $p < q$ be primes, $q \equiv 1 \pmod{p}$. Show there exists a unique non-abelian group of order pq.

Exercise 9. Let $\psi \in Aut(H)$. Show that for $\varphi' : g \mapsto \psi \varphi_g \psi^{-1}$ is $H \rtimes_\varphi K \cong H \rtimes_{\varphi'} K$.