TUTORIAL FOR THE SUBJECT NMAG336
INTRODUCTION TO THE CATEGORY THEORY

TUTORIAL 1 / FEBRUARY 24 2023

Problem 1.1. *Show that by composing two monomorphisms (or epimorphisms, sections, retractions, isomorphisms) we get again monomorphism (or epimorphism, section, retraction, isomorphism).*

Problem 1.2. *Consider the category* **Rel** *whose objects are sets and relation morphisms (with standard composition of relations).*

1. *Show that* $\varrho \in \mathbf{Rel}(A, B)$ *is a monomorphism if and only if there exists* $B' \subseteq B$ *such that* $\rho \cap (A \times B')$ *is a bijection* $A \to B'$.
2. *Decide when* $\rho \in \mathbf{Rel}(A, B)$ *is an epimorphism, or isomorphism.*
3. *Show that monomorphisms are sections and retraction are epimorphisms in the category* **Rel**.

Problem 1.3. *Let* $(P, \leq)$ *be a partially ordered set. Consider a category* $\mathbf{P}$ *whose objects are elements of the set* P*and morphisms correspond to ordered pairs of elements of* P. *This means that there is a morphism between a pair of objects* $p, q \in \mathbf{P}$ *if and only if* $p \leq q$ *and this morphism is unique.*

1. *Show that* $\mathbf{P}$ *is a category.*
2. *Show that every morphism in* $\mathbf{P}$ *is a bimorphism (i.e., a monomorphism and at the same time epimorphism).*
3. *Show that for a morphism* $f \in \mathbf{P}$ *is equivalent*
   - (i) f *is a section;*
   - (ii) g *is a retraction;*
   - (iii) f *is an isomorphism;*
   - (iv) f *is an identity morphism (on some object);*
4. *Let* $(Q, \leq)$ *be another ordered set and let* $\mathbf{Q}$ *be the corresponding category. Show that functors* $\mathbf{P} \to \mathbf{Q}$ *correspond to monotone (=order preserving) mappings* $P \to Q$.

Problem 1.4. *Let* **Pos** *denote the category of all partially ordered sets. Morphisms in this category are monotone mappings. Decide whether all monomorphisms in this category are sections and whether all bimorphisms are isomorphisms.*

Problem 1.5. *Let* **Grp** *denote the category of all groups (i.e., objects in this category are groups and morphisms are group homomorphisms).*

1. *Show that monomorphisms in* **Grp** *are precisely one-to-one homomorphisms.*
2. *Show that epimorphisms in* **Grp** *are precisely homomorphisms onto.*
3. *Describe the sections and retractions in this category.*
4. *Decide whether all bimorphisms in this category are isomorphisms.*

Problem 1.6. *Find a monoid (= a category with one object) that contains a bimorphism that is not an isomorphism. Can this monoid be finite?*

Problem 1.7. *Show that a full-faithful functor maps on a section only a section. Find an examples that none of the assumptions of fullness and faithfulness can be omitted.*

Problem 1.8. *Find an example of a functor whose image is not a subcategory (of the target category).*