TUTORIAL FOR THE SUBJECT NMAG336
INTRODUCTION TO THE CATEGORY THEORY

TUTORIAL 3 / MARCH 24 2023

Problem 3.1. *Let* $U\colon \mathbf{Vec}_T \to \mathbf{Set}$ *be the forgetful functor. For the set* X*, let* V_X *denote the arithmetic vector space with basis* X *(of all formal linear combinations of elements of* X*). Let* $u\colon X \to V_X$ *denote the inclusion map. Show that the pair* $\langle u, V_X\rangle$ *is a* universal morphism *from* X *to the forgetful functor* U.

Problem 3.2. *Let* **ID** *denote the category whose objects are integral domains and whose morphisms are one-to-one ring homomorphisms. Let* **Fld** *denote the category of all fields. Note that* **Fld** *is a complete subcategory of the category* **ID**. *Let* $U\colon \mathbf{Fld} \to \mathbf{ID}$ *be the forgetful functor that assigns to the field* T *the integral domain* $U(T)$*, formed by forgetting the partial unary operation of inversion (i.e., the operation* $t \mapsto t^{-1}$ *defined for* $t \in T \setminus \{0\}$*), and which is an identity on morphisms. Find a universal morphism from the integral domain* R *to the functor* U.

Problem 3.3. *Let* **A** *be a category and* $G\colon \mathbf{A} \to \mathbf{Set}$ *a functor. An* universal object *(sometimes called an* universal pair*) of the functor* G *is a pair* $\langle a, u\rangle$ *consisting of an object of the category* **A** *and an element* $u \in G(a)$ *such that*

> *for every pair* $\langle b, v\rangle$*, where* b *is an object of the category* A *and* $v \in G(b)$*, there is a unique morphism* $g\colon a \to b$ *in catA such that* $v = G(g)(u)$.

Given a set X *and an element* $x \in X$*, we denote by* $\dot{x}\colon \{\emptyset\} \to X$ *the map given by* $\dot{x}(\emptyset) = x$. *Prove that a pair* $\langle a, u\rangle$ *is a universal object of the functor* G *if and only if the pair* $\langle a, \dot{u}\rangle$ *is a universal morphism from the set* $\{\emptyset\}$ *into* G.

Problem 3.4. *Let* $F\colon \mathbf{A} \to \mathbf{B}$ *be a functor and let* b *be an object of a category* **B**. *Let* G *denote the functor* $G := \mathbf{B}(b, F(-)) = \mathbf{B}(b, -) \circ F\colon \mathbf{A} \to \mathbf{Set}$. *Show that the pair* $\langle a, u\rangle$ *is a universal morphism from* b *to* F *if and only if it is a universal object of the functor* G.

Problem 3.5. *What are products and co-products in the following catgories? The category*

- **Set** *of all sets;*
- $\mathbf{Vec}_T$ *of all vector spaces over the solid* T*;*
- **Grp** *of all groups;*
- **Ab** *of all abelian groups;*

Problem 3.6. *Let* P *be a partially ordered set viewed as a category. What are products and co-products in the category* P*? Characterize posets* P *that are complete (resp. co-complete) categories.*

Problem 3.7. *prove that every equalizer is a monomorphism and every co-equalizer is an epimorphism.*

Problem 3.8. *Characterize equalizers and co-equalizers in the category*

- **Set** *of all sets;*
- $\mathbf{Vec}_T$ *of all vector spaces over the solid* T*;*
- **Grp** *of all groups;*
- **Ab** *of all abelian groups;*