TUTORIAL FOR THE SUBJECT NMAG336
INTRODUCTION TO THE CATEGORY THEORY

TUTORIAL 4 / APRIL 21 2023

Problem 4.1. *Let A, resp., B, be the product of two copies of the abelian group $\mathbb{Z}_2$ in the category* **Ab** *of abelian groups, resp., in the category* **Grp** *of groups. Show that $A \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and that the group B is infinite.*

Problem 4.2. *Find the equalizer and coequalizer of homomorphisms*

$$\begin{aligned} f\colon \mathbb{Z} &\to \mathbb{Z}_{60} \\ a &\mapsto 6a \mod 60 \end{aligned}$$

and

$$\begin{aligned} g\colon \mathbb{Z} &\to \mathbb{Z}_{60} \\ a &\mapsto 10a \mod 60 \end{aligned}$$

in the category **Ab**.

Problem 4.3. *Let p be a prime number and*

$$\begin{aligned} f_{m,n}\colon \mathbb{Z}_{p^m} &\to \mathbb{Z}_{p^n} \\ a &\mapsto p^{n-m}a \end{aligned}$$

be a homomorphism for all pairs $m \leq n$ of positive integers.

Let **N** *denote the category induced by the set of natural numbers with an usual order. Objects of this category are natural numbers $1, 2, \dots$ and morphisms are ordered pairs of natural numbers. Consider the functor $F\colon \mathbf{N} \to \mathbf{Ab}$ given by*

- *$F(n) = \mathbb{Z}_{p^n}$ for every natural number;*
- *$F(m \leq n) = f_{m,n}$ for every ordered pair $m \leq n$ of natural numbers.*

Let $A = \bigoplus_{i \in \mathbb{N}} A_i$, where $A_i = \langle a_i \rangle$ is the infinite cyclic group generated by the element a_i, for each $i \in \mathbb{N}$. Let $a_0 = 0$ and denote by B the subgroup of the group A generated by the set $\{p \cdot a_i - a_{i-1} \mid i \in \mathbb{N}\}$. Put $L = A/B$ and for $n \in \mathbb{N}$ define a mapping

$$\begin{aligned} f_n\colon \mathbb{Z}_{p^n} &\to L \\ x &\mapsto x \cdot a_n. \end{aligned}$$

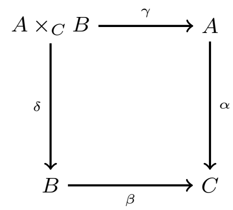
Prove that f_n are well-defined homomorphisms, that they are one-to-one, and that $\langle L \mid f_n; n \in \mathbb{N} \rangle$ is a limit of the diagram F.

Problem 4.4. *Let L_n denote the image of f_n in L. Show that*

1. *the order of each element of the group L is a power of the prime number p;*
2. *$L_n = \{x \in L \mid \text{the order of } x \text{ is at most } p^n\}$;*
3. *$\{0\} \subsetneq L_1 \subsetneq L_2 \subsetneq \dots$, and that L_n are only proper nontrivial subgroups of L;*
4. *$L \simeq L/L_n$ for every $n \in \mathbb{N}$, i.e. the group L is isomorphic to each of its factors.*

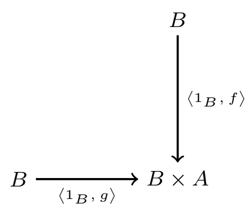
Problem 4.5. *Prove that the equation $a = n \cdot x$ with a variable x has a solution in the group L for every nonzero integer n.*

Problem 4.6. *Consider the pull-back $\langle A \times_C B \mid \delta, \gamma \rangle$ diagram $B \overset{\beta}{\to} C \overset{\alpha}{\leftarrow} A$ represented by the following diagram:*

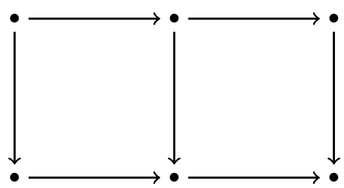


Prove that if β is a monomorphism, then γ is also a monomorphism.

Problem 4.7. *Prove that the equalizer of morphisms $f, g \colon B \to A$ can be constructed using the pull-back diagram*



Problem 4.8. *Consider a commutative diagram*



Prove that

1. *if both inner squares are pull-backs, the outer rectangle is also a pull-back;*
2. *if the outer rectangle and the right inner square are pull-backs, the left inner square is also a pull-back.*