TUTORIAL FOR THE SUBJECT NMAG336
INTRODUCTION TO THE CATEGORY THEORY

TUTORIAL 5 / MAY 5 2023

Problem 5.1. *The* diagonal functor $\Delta\colon \mathbf{A} \to \mathbf{A} \times \mathbf{A}$ *assigns*

- *to an object a of the category $\mathbf{A}$ the pair $\Delta(a) = \langle a, a\rangle$;*
- *to a morphism f in the category $\mathbf{A}$ the morphism $\Delta(f) = \langle f, f\rangle$.*

Let $\eta = \langle i, j\rangle : \langle a, b\rangle \to \Delta(c) = \langle c, c\rangle$ be a universal morphism from the object $\langle a, b\rangle$ to the diagonal functor Δ. Prove that $\langle c \mid i, j\rangle$ is a co-product of objects a, b in the category $\mathbf{A}$.

Problem 5.2. *Let $\mathbf{J}$ and $\mathbf{A}$ be categories. Let $\mathbf{A}^{\mathbf{J}}$ denote the category of all functors $\mathbf{J} \to \mathbf{A}$. The* diagonal functor $\Delta\colon \mathbf{A} \to \mathbf{A}^{\mathbf{J}}$ *assigns*

- *to an object a of category $\mathbf{A}$ the* constant functor *that maps every object of category $\mathbf{J}$ to a and every morphism of category $\mathbf{J}$ to the identity morphism 1_a;*
- *to a morphism $f\colon a \to b$ in $\mathbf{A}$ the natural transformation $\Delta(f)\colon \Delta(a) \to \Delta(b)$ given by $\Delta(f)_j = f$ for each $j \in \mathbf{ob}\,\mathbf{J}$.*

Let $F \in \mathbf{A}^{\mathbf{J}}$ be a diagram in the category $\mathbf{A}$ indexed by the category $\mathbf{J}$ and $\langle c, \eta\rangle$ be a universal morphism from F to Δ. Prove that $\langle c \mid \eta_j,\ j \in \mathbf{ob}\,\mathbf{J}\rangle$ is a colimit of the diagram F.

Problem 5.3. *Let $\mathbf{J}$, $\mathbf{A}$ be the categories and $\Delta\colon \mathbf{A} \to \mathbf{A}^{\mathbf{J}}$ be the functor as in the previous problem. Let $F \in \mathbf{A}^{\mathbf{J}}$ be a $\mathbf{J}$-indexed diagram in $\mathbf{A}$. Let $\langle d, \pi\rangle$ be a universal morphism from Δ to F. Prove that $\langle d \mid \pi_j,\ j \in \mathbf{ob}\,\mathbf{J}\rangle$ is a limit of the diagram F.*

Problem 5.4. *By a* representation *of a functor $F\colon \mathbf{A} \to \mathbf{Set}$ we mean the pair $\langle a, \varphi\rangle$, where $a \in \mathbf{ob}\,\mathbf{A}$ and $\varphi\colon A(a, -) \to F$ is a natural isomorphism. Let $F, G\colon \mathbf{A} \to \mathbf{Set}$ be functors with representations $\langle a, \varphi\rangle$, of the functor F, and $\langle b, \psi\rangle$, of the functor G. Prove that for every natural transformation $\tau\colon F \to G$ there exists a unique morphism $h\colon b \to a$ in the category $\mathbf{A}$ such that*

$$\tau \circ \varphi = \psi \circ \mathbf{A}(h, -)\colon \mathbf{A}(a, -) \to G.$$

Problem 5.5. *Let $\mathbf{A}$ be a complete subcategory of the category $\mathbf{B}$. Let $J\colon \mathbf{A} \to \mathbf{B}$ denote the inclusion functor. Prove that for every pair of functors $F, G\colon \mathbf{C} \to \mathbf{A}$:*

$$\mathbf{Nat}(F, G) \simeq \mathbf{Nat}(JF, JG).$$

Problem 5.6. *Prove that there exists a coproduct of objects a, b of a category $\mathbf{A}$ if and only if the functor*

$$\begin{aligned} \mathbf{A}(a, -) \times \mathbf{A}(b, -)\colon \mathbf{A} &\to \mathbf{Set} \\ c &\mapsto \mathbf{A}(a, c) \times \mathbf{A}(b, c) \\ f &\mapsto \mathbf{A}(a, f) \times \mathbf{A}(b, f) \end{aligned}$$

is representable.