TUTORIAL FOR THE SUBJECT NMAG336
INTRODUCTION TO THE CATEGORY THEORY

TUTORIAL 6 / MAY 19 2023

Problem 6.1. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be ordered sets considered as categories. Let* $F\colon \mathbf{P} \to \mathbf{Q}$ *and* $G\colon \mathbf{Q} \to \mathbf{P}$ *be monotone (ie, order-preserving) mappings. Show that the mapping* F *is left adjoint to* G *if and only if*

$$p \leq G(q) \iff F(p) \leq q$$

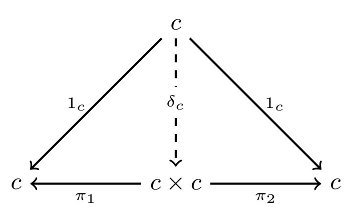
for all $p \in \mathbf{P}$ *and* $q \in \mathbf{Q}$.

Problem 6.2. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be ordered sets considered as categories. Suppose that there are arbitrary suprema and infima in* $\mathbf{P}$ *and* $\mathbf{Q}$. *Let* $F\colon \mathbf{P} \to \mathbf{Q}$ *be a suprema-preserving mapping. Show that there is a left adjoint to* F *and find it.*

Problem 6.3. *Let us have the situation as in Problem 6.1 and put* $H = GF\colon \mathbf{P} \to \mathbf{P}$. *Show that the mapping* H *is monotone, idempotent (ie, that* $H = HH$*), and that* $p \leq H(p)$ *for all* $p \in \mathbf{P}$. *What properties does the mapping* $K = FG\colon \mathbf{Q} \to \mathbf{Q}$ *satisfy?*

Problem 6.4. *Show that the product* $\times\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ *is a right adjoint to the diagonal functor* $\Delta\colon \mathbf{C} \to \mathbf{C} \times \mathbf{C}$.

Problem 6.5. *Show that the unit* $\langle \Delta, \times, \varphi \rangle$ *of the above adjunction is a natural transformation* $\delta\colon I_{\mathbf{C}} \to \times \circ \Delta$ *such that for each object* c *of the category* $\mathbf{C}$ *there is* $\delta_c\colon c \to c \times c$, *a (uniquely determined) morphism, such that the diagram*



commutes.

Problem 6.6. *Find a left adjoint to the diagonal functor* $\Delta\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$

Problem 6.7. *Let* R *be a ring. Find left and right adjoints to the forgetful functor* $U\colon R-\mathbf{Mod} \to \mathbf{Ab}$.

Problem 6.8. *Prove that torsion-free abelian groups form a reflexive subcategory of the category of all abelian groups.*