ABELIAN GROUPS WITH A MINIMAL GENERATING SET

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Abstract. We study the existence of minimal generating sets in abelian groups. We prove that abelian groups with minimal generating sets are closed neither under quotients, nor under subgroups, nor under infinite products. We give necessary and sufficient conditions for existence of a minimal generating set providing that the abelian group is uncountable, torsion, or torsion-free completely decomposable.

Introduction

Given a subset X of an algebra A, we denote by $\mathrm{Span}(X)$ a subalgebra of A generated by X. A set X is called $\mathcal{S}$-*independent* (see [3, Ch 1. §9]) if $x \notin \mathrm{Span}(X \setminus \{x\})$ for all $x \in X$. An $\mathcal{S}$-independent set generating A is called a *minimal generating set of* A.

Minimal generating sets of groups, rings, and fields were studied in [2] and [5]. In this paper, we will target the minimal generating sets of abelian groups.

It is easy to observe that nontrivial divisible abelian groups do not have minimal generating sets. On the other hand, we will prove that a direct sum of a divisible abelian group D and any abelian group with a minimal generating set of cardinality at least $\mathrm{card}(D)$ possesses a minimal generating set. Applying this observation, we will characterize uncountable abelian groups with a minimal generating set.

Next, we will prove that the class of abelian groups with a minimal generating set is closed neither under subgroups, nor under quotients, nor under infinite direct products.

Further, we will give a necessary and sufficient conditions for a torsion abelian group or a completely decomposable torsion-free abelian group to have a minimal generating set.

Given an abelian group A, we denote by $\mathrm{gen}(A)$ the minimal size of its generating set. By $\mathbb{P}$ we denote the set of all prime numbers.

1. Divisible Abelian Groups

Recall that an abelian group A is *divisible* if $nA = A$ for every integer n. Divisible abelian groups coincides with injective ones [6, Theorem 10.23] and each divisible abelian group is a direct sums of abelian groups isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}(p^\infty)$, for $p \in \mathbb{P}$ [6, Theorem 10.28].

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Theorem 1.1. *A nontrivial divisible abelian group does not have a minimal generating set.*

Proof. Every nontrivial divisible abelian group is infinitely generated. Suppose that there is a nontrivial divisible group D with a minimal generating set, say X. Let x be an element of X and denote by D' the subgroup of D generated by $X \setminus \{x\}$. Since X is a minimal generating set, $x \notin D'$ and D/D' is a cyclic group generated by the image of x. But a quotient of a divisible abelian group is divisible [6, Example 10.6], whence infinitely generated, which is not the case. □

Corollary 1.2. *The group $\mathbb{Q}$ of rational numbers as well as any product and any direct sum of copies of this group does not have a minimal generating set.*

We strengthen Theorem 1.1 as follows.

Lemma 1.3. *Let A be an abelian group and B a subgroup of A such that A/B is a non-trivial divisible abelian group. If $\mathrm{gen}(B) < \mathrm{card}(A)$, then A does not have a minimal generating set.*

Proof. Suppose that A has a minimal generating set X. Since A/B is a non-trivial divisible group, it is not finitely generated. It follows that A is not finitely generated and so X is infinite with $\mathrm{card}(X) = \mathrm{card}(A)$. Since $\mathrm{gen}(B) < \mathrm{card}(A)$, there is a proper subset F of X (of size less than $\mathrm{card}(A)$) such that $B \subseteq \mathrm{Span}(F)$. It follows that $A/\mathrm{Span}(F)$ is a non-trivial divisible group with a minimal generating set $\{x + \mathrm{Span}(F) \mid x \in X \setminus F\}$. This is a contradiction with Theorem 1.1. □

Lemma 1.4. *Let A be an abelian group and let F be a free abelian group with a free basis X. If $\mathrm{card}(X) \geq \mathrm{card}(A)$, then $F \oplus A$ has a minimal generating set.*

Proof. Since $\mathrm{card}(X) \geq \mathrm{card}(A)$, there is a surjective map $\pi : X \to A$. We claim that $Y = \{2x + \pi(x) \mid x \in X\} \cup \{3x + \pi(x) \mid x \in X\}$ is a minimal generating set of $F \oplus A$. It is easy to see that $\{2x \mid x \in X\} \cup \{3x \mid x \in X\}$ is a minimal generating set of F, whence the set Y is $\mathcal{S}$-independent. On the other hand

$$x = (3x + \pi(x)) - (2x + \pi(x))$$

for every $x \in X$, hence $X \subset \mathrm{Span}(Y)$. It follows that $A = \pi(X) \subseteq \mathrm{Span}(Y)$, whence $F \oplus A = \mathrm{Span}(Y)$. □

As opposed to Lemma 1.3, we prove that

Proposition 1.5. *Let D be an abelian divisible group and let A be an abelian group with a minimal generating. Then the direct sum $A \oplus D$ has a minimal generating set if and only if $\mathrm{gen}(A) \geq \mathrm{card}(D)$.*

Proof. The "only if" part follows from Lemma 1.3. ($\Leftarrow$) We can suppose that D is nontrivial, hence infinite. Let X be a minimal generating set of A, and let $\pi : \mathbb{Z}^{(X)} \to A$ be the projection extending identity on X. A subgroup of a free abelian group is free [6, Theorem 10.18], and so the kernel, K, of π is a free abelian group. If $\mathrm{gen}(K) < \mathrm{card}(D)$, then by the assumption $\mathrm{gen}(K) < \mathrm{gen}(A)$, and both A and $A \oplus D$ have a free direct summand of rank $\mathrm{card}(A \oplus D)$. In this case we apply Lemma 1.4. Suppose that $\mathrm{gen}(K) \geq \mathrm{card}(D)$. It follows that there is a projection $\varphi\colon K \to D$. Since divisible abelian groups correspond to injective abelian groups [6, Theorem 10.23], φ extends to a projection $\psi : \mathbb{Z}^{(X)} \to D$ (see Figure 1). We claim that $Y = \{\pi(x) + \psi(x) \mid x \in X\}$ is a minimal generating set of $A \oplus D$. Since its image in the quotient $(A \oplus D)/D \simeq A$ corresponds to the set X, the set Y is $\mathcal{S}$-independent. Let $d \in D$ and let $k \in K$ satisfy $\varphi(k) = d$. There are $x_1, x_2, \dots, x_n \in F$, and integers $z_1, z_2, \dots, z_n$ such that

$$k = \sum_{i=1}^{n} z_i x_i.$$

It follows that

$$\sum_{i=1}^{n} z_i(\pi(x_i) + \psi(x_i)) = \pi(k) + d = d,$$

whence $D \subset \mathrm{Span}(Y)$. Clearly then also $X \subseteq \mathrm{Span}(Y)$, and so the set Y generates $A \oplus D$. □

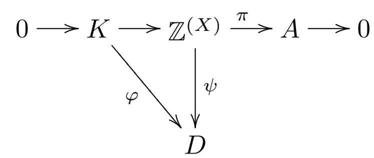

Figure 1

2. Subgroups and Products

It is clear that a direct sum of abelian groups with minimal generating sets have a minimal generating set as well. On the other hand, since every abelian group is a quotient of a free abelian group and divisible groups does not have a minimal generating sets, the abelian groups with minimal generating sets are not closed under quotients. It follows from Proposition 1.5 that a direct summand of an abelian group with a minimal generating set may not have a minimal generating set, in particular, abelian groups with minimal generating sets are not closed under subgroups. Not so obvious is that abelian groups with minimal generating sets are not closed under products, which is the aim of this section.

We begin with a lemma similar to Lemma 1.4, characterizing a wide class of groups with a minimal generating set, in particular, it implies that a product of copies of $\mathbb{Z}$ has a minimal generating set. The product of infinitely many copies of $\mathbb{Z}$ is a torsion free not completely decomposable group [4, p. 216], and the existence of a minimal generating set of this group cannot be decided applying results obtained in Sections 4 and 5.

Lemma 2.1. *Let A be an abelian group. If the dimensions of A/pA, and A/qA (as $\mathbb{Z}_p$, and $\mathbb{Z}_q$ vector space, respectively) are the same as* $\mathrm{gen}(A)$*, for two different prime numbers p, q, then A possesses a minimal generating set.*

Proof. Since p, q are relatively prime, there exist integers s, t such that $sp + tq = 1$. It follows that

$$A/pqA \simeq A/pA \oplus A/qA, \tag{2.1}$$

indeed, $a = spa + sqa$, for every $a \in A$, that is, $A = pA + qA$, and if $a = pb = qc$, for some $a \in A$, and b, $c \in A$, then $a = pq(sc + tb)$, that is, $pA \cap qA = pqA$. Let X be a generating set of A of the smallest cardinality. By our assumption,

$A/pA \simeq \mathbb{Z}_p^{(X)}$, and $A/qA \simeq \mathbb{Z}_q^{(X)}$. By (2.1), $A/pqA \simeq \mathbb{Z}_p^{(X)} \oplus \mathbb{Z}_q^{(X)} \simeq \mathbb{Z}_{pq}^{(X)}$, and so A/pqA possesses a minimal generating set $\{y_x + pqA \mid x \in X\}$, where $y_x + pqA$ are elements of order pq.

We claim that $Y = \{spy_x + pqx \mid x \in X\} \cup \{tqy_x - pqx \mid x \in X\}$ is a minimal generating set of A. It is easy to see that the image of Y in A/pqA forms a minimal generating set of the quotient, hence Y is $\mathcal{S}$-independent. Since

$$y_x = spy_x + pqx + tqy_x - pqx,$$

$y_x \in \mathrm{Span}(Y)$, for every $x \in X$, and so also $\{pqx \mid x \in X\} \subseteq \mathrm{Span}(Y)$, hence $pqA \subseteq \mathrm{Span}(Y)$, whence $A = \mathrm{Span}(Y)$. □

Corollary 2.2. *A products of copies of $\mathbb{Z}$ has a minimal generating set.*

Proof. Let $P = \mathbb{Z}^{\varkappa}$ be a product of $\varkappa$ copies of $\mathbb{Z}$. If $\varkappa$ is finite, the product is finitely generated, and so it has a minimal generating set. If $\varkappa$ is infinite, then $P/pP \simeq \mathbb{Z}_p^{\varkappa}$ has the same cardinality as P, for every prime number p, and we apply Lemma 2.1. □

It follows from Proposition 1.5 or Lemma 2.1 that $\mathbb{Q} \oplus \mathbb{Z}^{(\aleph_0)}$ has a minimal generating set. On the other hand, by Lemma 1.3, the group $\mathbb{Q} \oplus F$ does not have a minimal generating set for every finitely generated abelian group F. Also observe that, by Proposition 1.5,

$$\mathbb{Q}^{(\aleph_0)} \oplus \mathbb{Z}^{(\aleph_0)} \simeq (\mathbb{Q} \oplus \mathbb{Z})^{(\aleph_0)}$$

has a minimal generating set while the direct sum $\mathbb{Q} \oplus \mathbb{Z}$ has not, and so there is an infinite direct sum of abelian groups without a minimal generating set which have a minimal generating set. Finally, we prove that abelian groups with a minimal generating set are not closed under infinite products.

Example 2.3. The group

$$P = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$$

does not have a minimal set of generators.

Proof. We will regard elements of P as infinite sequences $(a_p)_{p \in \mathbb{P}}$, where $a_p \in \mathbb{Z}_p$. Denote by P' the subgroup of P generated by all sequences which are zero for all but finitely many prime numbers. Observe that for every prime p

$$pP = \{(a_q)_{q \in \mathbb{P}} \in P \mid a_p = 0\}.$$

It follows that $p(P/P') = P/P'$, for every prime number p, whence the group P/P' is divisible (in fact it is isomorphic to $\mathbb{Q}^{(\mathfrak{c})}$). Since P' is countably generated while the group P is uncountable, P does not have a minimal set of generators by Lemma 1.3. □

Corollary 2.4. *Groups with a minimal set of generators are not closed under infinite products.*

3. Uncountable Abelian Groups

Theorem 3.1. *An uncountable abelian group A has a minimal generating set if and only if there is a prime number p such that $\dim_{\mathbb{Z}_p}(A/pA) = \mathrm{card}(A)$.*

Proof. ($\Rightarrow$) Suppose that $\dim_{\mathbb{Z}_p}(A/pA) < \mathrm{card}(A)$, for every prime number p. Let for all $p \in \mathbb{P}$, B_p be a subset of A such that $\{b + pA \mid b \in B_p\}$ is a basis of A/pA. Set $B = \mathrm{Span}(\bigcup_{p\in\mathbb{P}} B_p)$. Then $\mathrm{card}(B) < \mathrm{card}(A)$, and A/B is divisible, which, by Lemma 1.3, implies that A does not have a minimal generating set.

($\Leftarrow$) Let A be an abelian group of an uncountable cardinality $\varkappa$, and suppose that $\dim_{\mathbb{Z}_p}(A/pA) = \mathrm{card}(A)$, for some prime number p. Let I_q, $q \in \mathbb{P}$, be pairwise disjoint sets such that $\dim_{\mathbb{Z}_q}(A/qA) = \mathrm{card}(I_q)$. For every prime number $q \neq p$, pick elements b_i, $i \in I_q$, in A so that $b_i + qA$, $i \in I_q$ form a basis of A/qA. Since the multiples $pb_i + qA$ form a basis of A/qA as well, we can assume that $b_i \in pA$, for every $i \in \bigcup_{q\neq p} I_q$. Set $I = \bigcup_{q\in\mathbb{P}} I_q$, and let $\{c_i + pA \mid i \in I\}$ be a basis of A/pA. Given a prime $q \neq p$, the elements c_i, $i \in I_q$, can be replaced by qc_i so that the new elements form a basis of A/pA. Thus we can without loss of generality assume that $c_i \in qA$ for every prime $q \neq p$ and every $i \in I_q$. Define elements a_i, $i \in I$ as follows: for $i \in I_p$ let $a_i = c_i$, for $i \in \bigcup_{q\neq p} I_q$ put $a_i = b_i + c_i$. Observe that $\{a_i + pA \mid i \in I\}$ is a basis of A/pA, hence the set $\{a_i \mid i \in I\}$ is $\mathcal{S}$-independent. Let B denote $\mathrm{Span}(a_i \mid i \in I)$. We claim that the abelian group $D = A/B$ is divisible. Suppose otherwise. Then there is a prime number q such that the abelian group

$$A/B\Big/q(A/B) \simeq A/qA\Big/(B + qA)/qA$$

is not trivial. First suppose that $q = p$. Then $c_i + pA \notin (B + pA)/pA$, for some $i \in I$. Since $b_i \in pA$, $a_i + pA = c_i + b_i + pA = c_i + pA$, for all $i \in \bigcup_{r\neq p} I_r$. But $a_i + pA \in B + pA$, for all $i \in I$, which is a contradiction. Now suppose that $q \neq p$. Then $b_i + qA \notin (B + qA)/qA$, for some $i \in I_q$. But $a_i + qA \in (B + qA)/qA$, and since, by our assumption $c_i \in qA$, also $b_i + qA = (a_i - c_i) + qA = a_i + qA$ belongs to $(B + qA)/qA$, which is again a contradiction.

Every divisible abelian group is a direct sum of its countable subgroups, hence there are countable subgroups D_α, $\alpha < \varkappa$, of A such that

$$D = \bigoplus_{\alpha<\varkappa} (D_\alpha + B)/B.$$

Since the groups $(D_\alpha + B)/B$ are divisible, $(D_\alpha + B)/B = p\big((D_\alpha + B)/B\big) = (pD_\alpha + B)/B$, and so we can assume that $D_\alpha \subseteq pA$, for every $\alpha < \varkappa$. By our assumption, $\mathrm{card}(I) = \varkappa$, thus we can identify the set I with the ordinal $\varkappa$.

For every $\lambda < \varkappa$ let B_λ denote $\mathrm{Span}(a_\beta \mid \beta < \lambda)$. For every $\alpha < \varkappa$ denote by λ_α the least ordinal such that $D_\beta \cap B = D_\beta \cap B_{\lambda_\alpha}$, for all $\beta \leq \alpha$. Notice that the groups $(D_\beta + B_{\lambda_\alpha})/B_{\lambda_\alpha}$ are divisible, and observe that

$$(D_\beta + B_{\lambda_\alpha})/B_{\lambda_\alpha} \simeq (D_\beta + B)/B,$$

for all $\beta \leq \alpha$. Let

$$\lambda_\alpha < \lambda_{\alpha,0} < \lambda_{\alpha,1} < \dots < \varkappa,$$

$\alpha < \varkappa$, be pairwise disjoint countable sequences. As in the proof of Proposition 1.5, there are elements $d_{\alpha,i}$, $i = 0, 1, \dots,$ of D_α such that $\{d_{\alpha,i} + a_{\lambda_{\alpha,i}} + B_{\lambda_\alpha} \mid i = 0, 1, \dots\}$ is a minimal generating set of the group

$$\overline{E_\alpha} = (D_\alpha + B_{\lambda_\alpha})\,/B_{\lambda_\alpha} \oplus \big(\big(\mathrm{Span}(a_{\lambda_{\alpha,i}} \mid i = 0, 1, \dots) + B_{\lambda_\alpha}\big)/B_{\lambda_\alpha}\big).$$

For all $\lambda < \varkappa$ define $e_\lambda \in A$ as follows:

$$e_\lambda = \begin{cases} d_{\alpha,i} + a_{\lambda_{\alpha,i}}, & \text{if } \lambda = \lambda_{\alpha,i} \text{ for some } \alpha < \varkappa, i \in \{0, 1, \dots\}; \\ a_\lambda, & \text{otherwise.} \end{cases}$$

We claim that $\{e_\lambda \mid \lambda < \varkappa\}$ is a minimal generating set of A. Since $e_\lambda + pA = a_\lambda + pA$, for every $\lambda < \varkappa$, the elements $e_\lambda + pA$ form a basis of A/pA, and so the set $\{e_\lambda \mid \lambda < \varkappa\}$ is $\mathcal{S}$-independent. Since the set $\{e_{\lambda_{\alpha,i}} \mid i = 0, 1, \dots\}$ generates $\overline{E_\alpha}$, $D_\alpha + B_{\lambda_\alpha} \subseteq E + B_{\lambda_\alpha}$, for every $\alpha < \varkappa$. We prove by induction that $D_\alpha \subseteq E$, for all $\alpha a < \varkappa$. Let $\alpha < \varkappa$, and suppose that $D_\beta \subseteq E$, for all $\beta < \alpha$. Then $a_\alpha \in E$, for every $\alpha < \lambda_\alpha$, that is, $B_{\lambda_\alpha} \subseteq E$. Since $D_\alpha + B_{\lambda_\alpha} \subseteq E + B_{\lambda_\alpha}$, we conclude that $D_\alpha \subseteq E$. It follows readily from the definition of the elements e_λ that $E + \bigcup_{\alpha<\varkappa} D_\alpha = A$, and since $\bigcup_{\alpha<\varkappa} D_\alpha \subseteq E$, we conclude that $E = A$. □

Applying this theorem, we can generalize Corollary 2.2.

Corollary 3.2. *Let A be a non-divisible abelian group of size λ. If $\varkappa$ is a cardinal number satisfying $\lambda^\varkappa = 2^\varkappa$, then $A^\varkappa$ has a minimal generating set. In particular, an infinite product of copies of a countable group A has a minimal generating set if and only if the group A is not divisible.*

Proof. Since A is not divisible, there is a prime number p such that A/pA is not trivial. It follows that

$$\dim_{\mathbb{Z}_p} A^\varkappa / pA^\varkappa \simeq \dim_{\mathbb{Z}_p} (A/pA)^\varkappa \geq 2^\varkappa = \lambda^\varkappa = \operatorname{card}(A^\varkappa).$$

For the case of products of countable groups, recall that divisible groups are closed under products. □

4. Torsion Abelian Groups.

Recall [6, p. 326] that a pure subgroup B of a torsion abelian group A is *basic* if it is isomorphic to the direct sum of cyclic groups, and the quotient A/B is divisible. By [6, Theorem 10.36] every torsion group has a basic subgroup. By [6, Theorem 10.40], any two basic subgroups of a p-primary abelian group are isomorphic. It follows that any two basic subgroups of a torsion abelian group are isomorphic, indeed, a p-primary component of a basic subgroup of a torsion abelian group is a basic subgroup of its p-primary component. Applying Lemma 1.3, we obtain the following proposition.

Proposition 4.1. *Let A be a torsion abelian group and B its basic subgroup. If* $\operatorname{card}(B) < \operatorname{card}(A)$*, then A does not have a minimal generating set.*

Lemma 4.2. *An uncountable torsion group A has a minimal generating set if and only if its basic subgroup has the same cardinality as the group A.*

Proof. The "only if" part again follows from Proposition 4.1. ($\Leftarrow$) Denote by λ the uncountable cardinality of the group A. Let B be its basic subgroup with $\operatorname{card}(B) = \lambda$. Then B is a direct sum of groups B_{p^n}, where p is prime and $n \in \mathbb{N}$, isomorphic to $\mathbb{Z}_{p^n}^{(\lambda_{p^n})}$. Since λ is uncountable, there exist a prime number p and $n \in \mathbb{N}$ with $\lambda_{p^n} = \lambda$. Then B_{p^n} is a pure subgroup of A of a bounded order, and so it is its direct summand [6, Corollary 10.41]. Let A' be a complement of B_p in A. Then

$$A/pA \simeq (A'/pA') \oplus (B_{p^n}/pB_{p^n}),$$

hence $\dim_{\mathbb{Z}_p}(A/pA) \geq \dim_{\mathbb{Z}_p}(B_{p^n}/pB_{p^n}) = \lambda_{p^n} = \lambda = \mathrm{card}(A)$. Now apply Theorem 3.1. □

Recall that an abelian group is *reduced* if it does not have a nontrivial divisible direct summand. Every abelian group is a direct sum of a reduced and a divisible abelian group. Countable reduced p-primary groups are classified by countable transfinite sequences of their *Ulm factors*. We make use of this classification to prove that every countable reduced torsion group has a minimal generating set.

Lemma 4.3. *Every countable reduced torsion group has a minimal generating set.*

Proof. Let A be a countable reduced torsion group. Since A is a direct sum of its p-primary components, we can without loss of generality assume that A is p-primary itself. Let τ be a type of A and $\{A_\alpha \mid 0 \leq \alpha < \tau\}$ its Ulm factors (see [4, pp. 189–190]). The group A contains a descending sequence

$$A = A'_0 \supset A'_1 \supset \cdots \supset A'_\alpha \supset \cdots \supset A'_\tau = 0$$

of length τ such that $A'_\tau = 0$ and $A'_\alpha/A'_{\alpha+1} \simeq A_\alpha$, for every $0 \leq \alpha < \tau$. By [4, Existence Theorem (p. 190)], each A_α is a countable direct sum of cyclic p-primary groups. As in the proof of the Existence theorem, denote by $a_{\alpha,1}, a_{\alpha,2}, \ldots$ generators of these cyclic groups, where $a_{\alpha,i}$ has order $p^{n_{\alpha,i}}$. Then the group A possesses generators $c_{\alpha,i}$, where $c_{\alpha,i}$ corresponds to $a_{\alpha,i}$, such that with each $c_{\alpha,i}$ is associated either an equation $p^{n_{\alpha,i}}c_{\alpha,i} = 0$, or an equation $p^{n_{\alpha,i}}c_{\alpha,i} = c_{\beta,j}$, where $\beta > \alpha$, and the following properties are satisfied:

1) There is no infinite sequence of relations
$$p^{n_{\alpha_0,i_0}}c_{\alpha_0,i_0} = c_{\alpha_1,i_1},\ p^{n_{\alpha_1,i_1}}c_{\alpha_1,i_1} = c_{\alpha_2,i_2},\ \ldots.$$
2) If $0 \leq \gamma < \beta$, and N is a natural number, then for every $c_{\beta,j}$, there exists an element $c_{\alpha,i}$ such that $\gamma \leq \alpha < \beta$, $N < n_{\alpha,i}$ and that relation associated with the element is $p^{n_{\alpha,i}}c_{\alpha,i} = c_{\beta,j}$.
3) If σ is a limit ordinal, then for every γ less than σ and every natural number N there exists an element $c_{\alpha,i}$ such that $\gamma < \alpha$, $N < n_{\alpha,i}$, and the relation associated with it is $p^{n_{\alpha,i}}c_{\alpha,i} = 0$.

There exists a system of equations satisfying these relations, and the abelian group defined by generators $c_{\alpha,i}$ and the relations is by Ulm's Theorem [4, p. 193] unique up to an isomorphism, hence isomorphic to A. It follows from property 2) that the set $X = \{c_{0,i} \mid i = 0, 1, \ldots\}$ generates A. Elements $c_{0,i}$ correspond to elements $a_{0,i}$ which form a minimal generating set of the group A_0 isomorphic to the quotient A/A'_1. Thus the set X is $\mathcal{S}$-independent, and so it is a minimal generating set of A. □

Now we are ready to characterize torsion abelian group with a minimal generating set:

Theorem 4.4. *Let A be an infinite torsion abelian group with a basic subgroup B. Then A has a minimal generating set if and only if* $\mathrm{card}(B) = \mathrm{card}(A)$.

Proof. The equivalence is clear when A finite. Lemma 4.2 corresponds to the uncountable case. Thus suppose that A is countable infinite. If $\mathrm{card}(B) < \mathrm{card}(A)$, then A does not have a minimal generating set by Lemma 1.3. Suppose that $\mathrm{card}(B) = \mathrm{card}(A)$. Let $A = D \oplus R$ be a decomposition of the group A into a direct sum of a divisible group D and a reduced group R [6, Theorem 10.26]. If R

was finitely generated, it would coincide with B which is not finitely generated by our assumption. Hence R has an infinite minimal generating set and we can apply Proposition 1.5. □

Corollary 4.5. *A torsion abelian group A has a minimal generating set if and only if there is a subgroup A' of A such that the quotient A/A' is isomorphic to a direct sum of cyclic groups and* $\mathrm{card}(A) = \mathrm{card}(A/A')$.

Proof. The corollary is clear for finite abelian groups, so suppose that A is infinite. ($\Leftarrow$) Denote by C the quotient A/A' and assume that $\mathrm{card}(A) = \mathrm{card}(C)$. Let B be a basic subgroup of A. Assume on the contrary that the group A does not have a minimal generating set, that is, by Theorem 4.4, that $\mathrm{card}(B) < \mathrm{card}(A)$. Denote by B' the quotient $(B + A')/A'$. Observe that

$$C/B' = A/A' \Big/ (A' + B)/A' \simeq A/B \Big/ (A' + B)/B$$

is a quotient of divisible group, whence itself divisible. Since $\mathrm{card}(B') \leq \mathrm{card}(B) < \mathrm{card}(A) = \mathrm{card}(C)$, the group C/B' has a cyclic direct summand, which is in the contrary with its divisibility.

($\Rightarrow$) Let A be a torsion group with a minimal generating set. Denote by A_p the p-primary components of A. The group A is a direct sum of its p-primary components by [6, Theorem 10.7], therefore it can by identified with $\bigoplus_{p\in\mathbb{P}} A_p$. Denote by B_p a basic subgroup of A_p. Then $\bigoplus_{p\in\mathbb{P}} B_p$ is a basic subgroup of A. Set $A' = \bigoplus_{p\in\mathbb{P}} pA_p$ and $B' = \bigoplus_{p\in\mathbb{P}} pB_p$. Then $A/A' \simeq B/B'$, and it is a direct sum of cyclic groups. We assume that A is infinite with a minimal generating set, thus, by Theorem 4.4, B is infinite of the same cardinality as A. It follows that $\mathrm{card}(B/B') = \mathrm{card}(A)$. □

In contrast to Lemma 4.3, an uncountable reduced p-primary group does not necessarily have a minimal generating set. This is proved in the last example of this section. Before recall that an element a of a p-primary abelian group is of infinite *height* provided that for every positive integer n, the equation

$$p^n x = a \tag{4.1}$$

has a solution. Notice that a p-primary abelian group without elements of infinite height is reduced.

Example 4.6. Let p be a prime number. The torsion subgroup T of the product $\prod_{n=1}^{\infty} \mathbb{Z}_{p^n}$ is reduced and does not have a minimal generating set.

Proof. We regard elements of T as infinite sequences $(a_0, a_1, \dots)$, where $a_n \in \mathbb{Z}_{p^n}$. We denote by B the subgroup of T of sequences which are finite for all but finitely many a_n. Then T corresponds to a closure of B in sense of [4, p. 184], and B is its basic subgroup. Since every element of T has a finite height, the abelian group is reduced. Since B is countable, while T is not, T does not have a minimal generating set. □

5. Completely Decomposable Groups

In this section we will investigate torsion-free abelian groups. Although we will not fully characterize countable torsion-free groups with a minimal generating set, we will prove some partial results for completely decomposable groups whose structure is well understood (see [1], [4, §§ 30, 31], and [6, pp. 331-335]). Recall that a *rank* of a torsion-free group is the size of its maximal independent subset (in the sense of [6, p. 127], which is stronger then the $\mathcal{S}$-independence).

A *p-height* of an element a of an abelian group A, denoted by $h_p(a)$, is the maximal nonnegative integer such that equation (4.1) has a solution in A. If the solution exists for every nonnegative integer, then we define $h_p(a) = \infty$. A *height sequence* of a is the sequence $h(a) = (h_2(a), h_3(a), \dots, h_p(a), \dots)$, where the indices run over all prime numbers. A characteristic is a sequence of nonnegative integers and the symbol ∞. Two characteristic are *equivalent*, if they differ in at most finite number of coordinates and they have ∞ in the same coordinates. Equivalence classes of characteristics are their *types*.

Abelian torsion free groups of rank 1 are exactly abelian groups isomorphic to subgroups of $\mathbb{Q}$ [6, p. 321]. The height sequences of any two nonzero elements of a torsion free abelian group A of rank 1 have the same types [6, Lemma 10.46], and so we can define a *type* of A as the type of the height sequence of any of its nonzero elements. Two torsion free abelian groups of the same type are isomorphic and each type is represented as the type of some torsion free abelian group of rank 1 [6, Theorems 10.47-48]. Moreover, a torsion free abelian group of rank 1 of a type τ contains an element a with $h(a) = h$, for any characteristic h of type τ (see the proof of [6, Theorem 10.48]).

A torsion free abelian group A is called *completely decomposable* if it is a direct sum of torsion free abelian groups of rank 1. Any two decompositions of a completely decomposable abelian group into the direct sum of groups of rank 1 are isomorphic [4, p. 212].

We are going to make use of a coarser classification of torsion free abelian groups of rank 1. Thus a torsion free abelian group A of rank 1 is of Type I if its type is represented by the characteristic $h = (h_2, h_3, \dots, h_p, \dots)$ with $0 < h_p < \infty$ for infinitely many primes p, and it is of Type II if it is neither divisible nor of Type I.

Lemma 5.1. *Let A be an abelian group and let F be its finitely generated subgroup. Then A has a minimal generating set if and only if A/F has a minimal generating set.*

Proof. First observe that if X, Y are subsets of an abelian group A such that both the image of X in $\operatorname{Span}(X)/\big(\operatorname{Span}(X) \cap \operatorname{Span}(Y)\big)$, and the image of Y in $\operatorname{Span}(Y)/\big(\operatorname{Span}(X) \cap \operatorname{Span}(Y)\big)$ are $\mathcal{S}$-independent, then $X \cup Y$ is a $\mathcal{S}$-independent subset of A.

($\Rightarrow$) Let X be a minimal generating set of A. Then $F \subseteq \operatorname{Span}(X_1)$ for some finite subset X_1 of X. Observe that the image of $X \smallsetminus X_1$ in $\operatorname{Span}(X \smallsetminus X_1)/\big(\operatorname{Span}(X_1) \cap \operatorname{Span}(X \smallsetminus X_1)\big)$ is $\mathcal{S}$-independent, the more so the image of $X \smallsetminus X_1$ in A/F is $\mathcal{S}$-independent. The quotient $\operatorname{Span}(X_1)/\big(\operatorname{Span}(X_1) \cap \operatorname{Span}(X \smallsetminus X_1)\big)$ is finitely generated, and so it possesses a minimal generating set Y'. Denote by Y the preimage of Y' in $\operatorname{Span}(X_1)/F$. Then the union of the image of $X \setminus X_1$ in A/F and Y forms a minimal generating set of A/F.

($\Leftarrow$) Let X' be a minimal generating set of A/F. Denote by X its preimage in A. Since F is finitely generated, the quotient $A/\operatorname{Span}(X) \simeq F/\big(F \cap \operatorname{Span}(X)\big)$ is finitely generated as well, and so it has a minimal generating set Y'. Denote by Y the preimage of Y, and conclude that $X \cup Y$ is a minimal generating set of A. $\square$

Theorem 5.2. *Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ be a finite direct sum of torsion-free abelian groups A_1, A_2, …, A_n of rank 1. Then A has a minimal generating set if and only if either A is free or one of the groups A_i is of Type I.*

Proof. Let x_i be a nonzero element of A_i. By Lemma 5.1, we can restrict ourselves to the quotient

$$A' = A/\mathrm{Span}(x_1, x_2, \dots, x_n) = \big(A_1/\mathrm{Span}(x_1)\big) \oplus \cdots \oplus \big(A_n/\mathrm{Span}(x_n)\big).$$

This is a countable torsion group isomorphic to a direct sum $D \oplus B$, where D is divisible and B is a direct sum of cyclic groups. By Theorem 4.4, the group A' has a minimal generating set if and only if either $\mathrm{card}(B) = \aleph_0$, which happens if and only if at least one of the groups A_i is of Type I, or B is finite and D is trivial which is equivalent to A being free. □

Lemma 5.3. *An infinite direct sum of torsion-free abelian groups of rank 1 of Type II has a minimal generating set.*

Proof. It suffices to prove the lemma for countable direct sums. Let $A = \bigoplus_{m,n} A_{m,n}$ be a countable direct sum of torsion-free abelian groups $A_{m,n}$ of rank 1 of Type II. Each of the abelian groups $A_{m,n}$ contains an element $a_{m,n}$ such that $h_{p_{m,n}}(a_{m,n}) = 1$ for some prime number $p_{m,n}$, and $h_q(a_{m,n}) = 0$, or $h_q(a_{m,n}) = \infty$ for every $q \neq p_{m,n}$. Let $b_{m,n} \in A$ be such that $a_{m,n} = p_{m,n} b_{m,n}$. Denote $A_i = \bigoplus_{n \leq i} \big(\bigoplus_m A_{m,n}\big)$. By induction we construct an increasing sequence $X_0 \subseteq X_1 \subseteq \dots$ such that X_i are $\mathcal{S}$-independent subsets of A_i, $A_i/\mathrm{Span}(X_i)$ are divisible, and $A_{i-1} \subseteq \mathrm{Span}(X_i)$ for all $1 \leq i$. Thus the union $X = \bigcup_{i=1}^{\infty} X_i$ will be a minimal generating set of A.

Set $X_0 = \{b_{0,n} \mid n = 0, 1, \dots\}$. Clearly X_0 is $\mathcal{S}$-independent, and observe that the abelian group $A_0/\mathrm{Span}(X_0)$ is divisible.

Suppose that the sequence is constructed up to some positive integer i. By the induction hypothesis, there is a subset X_i of A_i with the required properties, that is, $A_i/\mathrm{Span}(X_i)$ are divisible, and if $1 \leq i$, then $A_{i-1} \subseteq \mathrm{Span}(X_i)$. By Proposition 1.5, the group $A'_i = \big(A_i/\mathrm{Span}(X_i)\big) \oplus \mathrm{Span}(b_{0,i+1}, b_{1,i+1}, \dots)$ has a minimal generating set Y_{i+1} whose image in

$$A'_i/\big(A_i/\mathrm{Span}(X_i)\big) \simeq \mathrm{Span}(b_{0,i+1}, b_{1,i+1}, \dots)$$

corresponds to $\{b_{0,i+1}, b_{1,i+1}, \dots\}$. It follows that the set $X_{i+1} = X_i \cup Y_{i+1}$ is $\mathcal{S}$-independent, and that $A_i \subseteq \mathrm{Span}(X_{i+1})$. Finally observe that the quotient

$$A_{i+1}/\mathrm{Span}(X_{i+1}) \simeq \big(\bigoplus_{m=0}^{\infty} A_{m,i+1}\big)/\mathrm{Span}(b_{0,i+1}, b_{1,i+1}, \dots)$$

is divisible. □

Lemma 5.4. *Every completely decomposable reduced torsion free abelian group of an infinite rank has a minimal generating set.*

Proof. Let A be a completely decomposable reduced torsion free abelian group of an infinite rank $\varkappa$. Than A does not have nontrivial divisible subgroup, and so it has a decomposition

$$A = \big(\bigoplus_{\lambda < \varkappa_1} A_\lambda\big) \oplus \big(\bigoplus_{\lambda < \varkappa_2} B_\lambda\big) \tag{5.1}$$

into a direct sum of torsion free abelian group of rank 1, where $\varkappa_1 + \varkappa_2 = \varkappa$, A_λ are torsion free abelian group of rank 1 of Type I, and B_λ are torsion free abelian group of rank 1 of Type II. Abelian groups A_λ have minimal generating sets by Theorem 5.2. If $\aleph_0 \le \varkappa_2$, then $\bigoplus_{\lambda<\varkappa_2} B_\lambda$ has a minimal generating set by Lemma 5.3, while if $\varkappa_2$ is finite, we can rewrite (5.1) as

$$A = \Big(\bigoplus_{1\le\lambda<\varkappa_1} A_\lambda\Big) \oplus \Big(A_0 \oplus \bigoplus_{\lambda<\varkappa_2} B_\lambda\Big),$$

and we apply Theorem 5.2. □

Theorem 5.5. *Let A be a completely decomposable torsion free abelian group of an infinite rank and let $A = R \oplus D$ be its decomposition into a direct sum of a divisible group D and a reduced abelian group R. Then A has a minimal generating set if and only if* $\operatorname{rank}(R) \ge \operatorname{card}(D)$.

Proof. It is a consequence of Lemma 5.4, and Proposition 1.5 providing that the abelian group R is completely decomposable. Let

$$A = \Big(\bigoplus_{\lambda<\varkappa_1} A_\lambda\Big) \oplus \Big(\bigoplus_{\lambda<\varkappa_2} D_\lambda\Big)$$

be a decomposition of A into a direct sum of torsion-free abelian groups of rank 1, where D_λ are divisible while A_λ are not. It suffices to prove that $D = \bigoplus_{\lambda<\varkappa_2} D_\lambda$, because then

$$R \simeq A/\Big(\bigoplus_{\lambda<\varkappa_2} D_\lambda\Big) \simeq \bigoplus_{\lambda<\varkappa_1} A_\lambda.$$

Suppose otherwise. Then

$$D/\Big(\Big(\bigoplus_{\lambda<\varkappa_2} D_\lambda\Big) \cap D\Big) \hookrightarrow A/\Big(\bigoplus_{\lambda<\varkappa_2} D_\lambda\Big) \simeq \bigoplus_{\lambda<\varkappa_1} A_\lambda,$$

and so the abelian group $\bigoplus_{\lambda<\varkappa_1} A_\lambda$ has a nontrivial divisible subgroup, which we denote by D'. For $\mu < \varkappa_1$ denote by $\pi_\mu : \bigoplus_{\lambda<\varkappa_1} A_\lambda \to A_\mu$ the canonical projection. There is $\mu < \varkappa_1$ such that $\pi_\mu(D')$ is a nontrivial subgroup of A_μ. Since a quotient of a divisible group is divisible [6, Example 10.6], A_μ has a nontrivial divisible subgroup. Since A_μ is of rank 1, it is indecomposable, and so it is divisible itself, which is not the case. □

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