Riemannian geometry II

Plan: 1) topology, geometry, differential operators
Laplace – Beltrami operator on Riemannian manifold
Laplace – Hodge operator

2) Thurston's Geometrization conjecture
about Riemannian structures on 3 – manifolds
implies Poincaré conjecture (Perelman)
Uniformization theorem for 2 – manifolds
Every closed Riemannian surface admits a Riemannian metric with constant Gauss (or scalar) curvature

3) ?

Introduction

1) Marc Kac : Can one hear the shape of a drum, 1966.

2146 citations

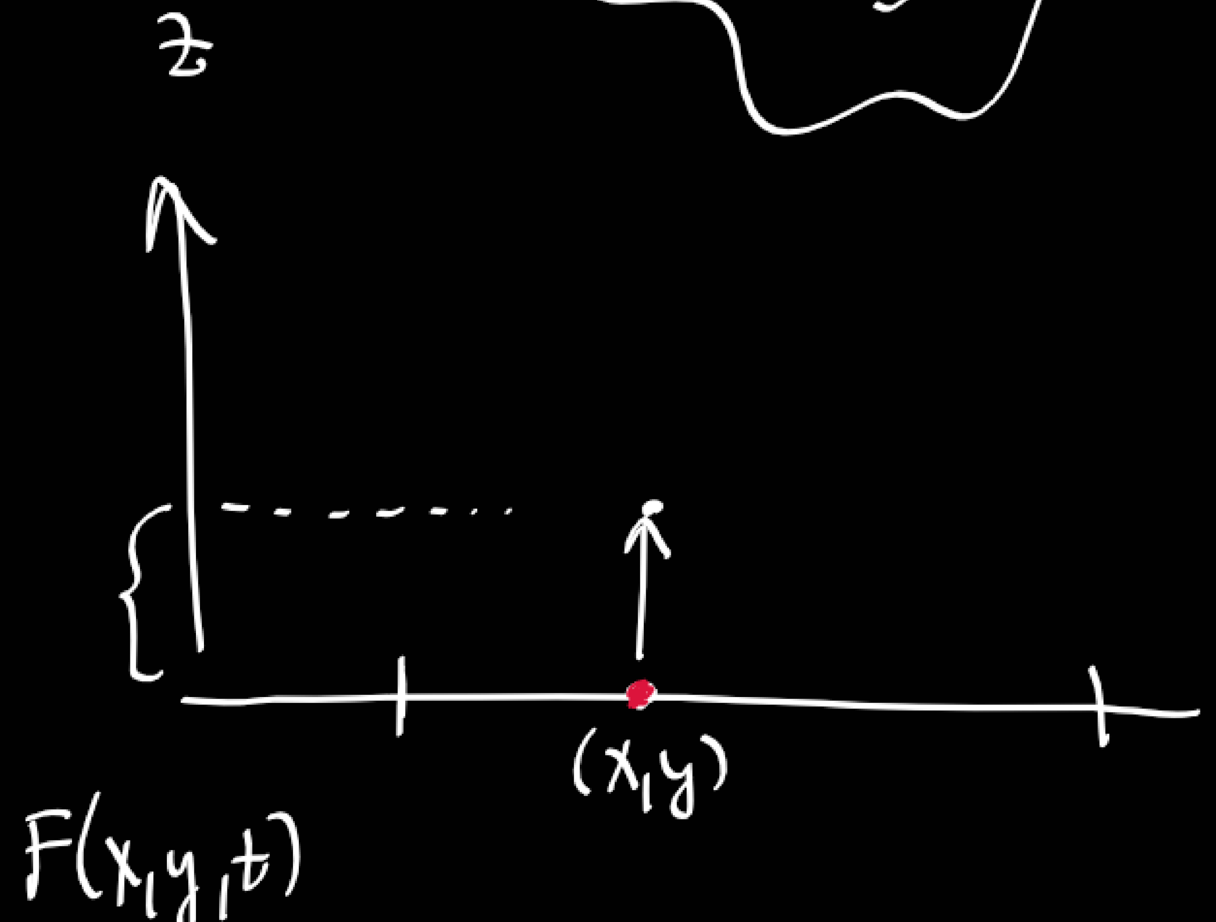
Problem: bounded region Ω in $\mathbb{R}^2 = \{z = 0\} \subseteq \mathbb{R}^3$
assume that $\partial\Omega$ is a Jordan curve Γ

There is a membrane attached to Γ
membrane is set in motion (perpendicular to the plane $\{z = 0\}$) then it obeys the equation

$$\frac{\partial^2 F}{\partial t^2} = c^2 \Delta F$$

↑ physical constant

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$



F is a function of x, y, t that describes the motion
$F(x,y,t)$ ... the height of the point of the membrane with x,y coordinates

$F = 0$ on $\partial\Omega$

We are interested in solutions

$$F : \overline{\Omega} \times \mathbb{R}^{\geq 0} \longrightarrow \mathbb{R}$$

wave equation in $\mathbb{R}^{2,1}$ $\quad \dfrac{\partial^2 F}{\partial t^2} = c^2 \Delta F \quad , \quad F = 0$ on $\partial\Omega$

Special solutions of this problem which are of interest

$$F(x,y,t) = u(x,y) \underbrace{e^{i\omega t}}$$

$$\operatorname{Re} e^{i\omega t} = \cos(t\omega)$$

harmonic tones or pure tones

$$\left.\begin{array}{l} \dfrac{\partial^2 F}{\partial t^2} = -\omega^2 F = -\omega^2 u e^{i\omega t} \\ \quad \parallel \\ c^2 \Delta F = c^2 (\Delta u) e^{i\omega t} \end{array}\right\} \quad -\omega^2 u = c^2 \Delta u$$

$(*)$ $\boxed{0 = \frac{1}{2} \Delta u + \omega^2 u}$ $\boxed{u = 0 \text{ on } \partial\Omega}$

$u \equiv u(x,y)$, $c^2 = \frac{1}{2}$

number ω is called a pure tone if there exist a non-trivial (non-zero) solution of $(*)$

pure tone is an eigenvalue of Δ

$$\boxed{\frac{1}{2} \Delta u = -\omega^2 u}$$

it can be shown that these pure tones form a discrete infinite sequence $\omega_1 \leq \omega_2 \leq \cdots\cdots$

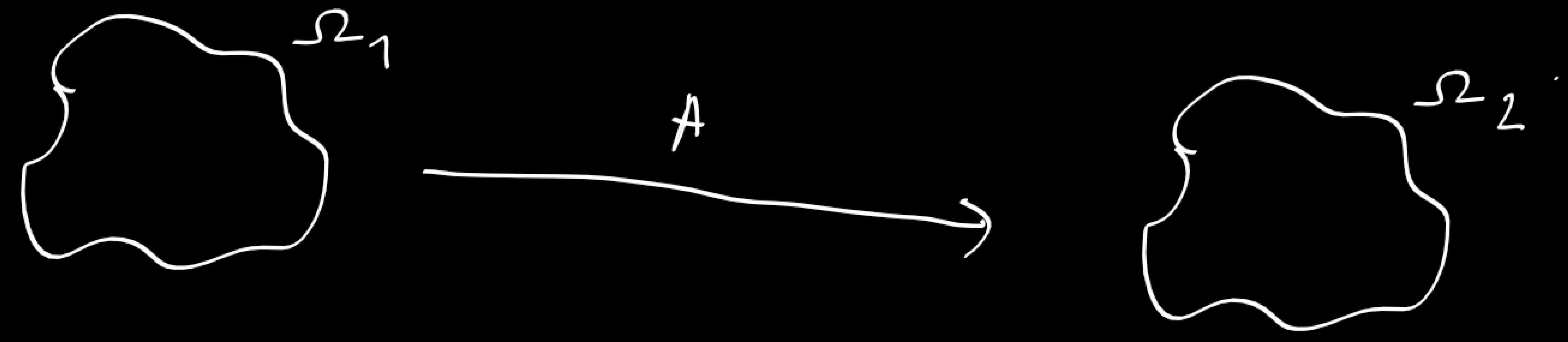
Assume now that we are given two regions

Ω_1 with $\partial\Omega_1 = \Gamma_1$ (Jordan curve) and

Ω_2 with $\partial\Omega_2 = \Gamma_2$

Assume that $\lambda_1 \leq \lambda_2 \leq \cdots$ pure tones for Ω_1

—"— $\mu_1 \leq \mu_2 \leq \cdots$ —"— Ω_2



Assume that these regions are congruent under an Euclidean motion, that is, there is $A \in Euc(2)$ (A is a composition and rotation) such that

$A(\Omega_1) = \Omega_2$, these two regions have the same shape

Now it follows that (since Δ in $\mathbb{R}^2$ is invariant $Euc(2)$)

if μ_i is a pure tone with function U_i on Ω_2 ($\frac{1}{2}\Delta U_i = -\mu_i^2 U_i$ on Ω_2), then

$$V_i = U_i \circ A : \Omega_1 \longrightarrow \mathbb{R} \quad \text{and} \quad \Delta V_i = -\mu_i^2 V_i \text{ on } \Omega_1$$

It follows that any pure tone on Ω_2 is also a pure tone on Ω_1, $\{\mu_i\}_{i\in\mathbb{N}} \subseteq \{\lambda_i\}_{i\in\mathbb{N}}$.

Reversing Ω_2 with Ω_1 and A with A^{-1}, then $\{\lambda_i\}_{i\in\mathbb{N}} \subseteq \{\mu_i\}_{i\in\mathbb{N}}$

For two congruent regions $\Rightarrow$ the sequences of pure tones agree.

Does the other implication hold? Can one hear a shape of a drum?

Not true, 1992, Gordon, Webb

One can hear the volume of a drum

Lebesgue measure of Ω $\rightarrow$ $\dfrac{|\Omega|}{2\pi} = \lim\limits_{r\to+\infty} \dfrac{N(r)}{r}$

$$N(r) = \#\{i \mid \lambda_i \le r\}$$

λ_i is a pure tone on Ω

Link between the spectral theory of elliptic differential operators on a manifold M and its geometric/topological properties

2) Assume that M is a smooth compact manifold of dim m

basic topological invariant is the Euler characteristic $\chi(M)$

if $m=2 \Rightarrow \chi(M)$ can be computed from the scalar curvature

$$\chi(M) = \underset{\substack{\uparrow \\ \text{i-th singular cohomology}}}{\sum_{i=0}^{m} (-1)^i H^i(M,\mathbb{R})} \underset{\substack{\uparrow \\ \text{de Rham theorem}}}{=} \sum_{i=0}^{m} (-1)^i H^i_{DR}(M) = \begin{array}{l}\text{index of} \\ \text{the Rham} \\ \text{complex}\end{array}$$

de Rham complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M)$$

assume that M is a topological, oriented, closed 4-manifolds

intersection form

$$Q_M : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \xrightarrow{\cup} H^4(M,\mathbb{Z}) \xrightarrow{\langle -, [M] \rangle} \mathbb{Z}$$

$$Q_M(a,b) = \langle a \cup b, \underset{\substack{\uparrow \\ \text{fundamental class of } M}}{[M]} \rangle \left(= \int_M \alpha \wedge \beta \right)$$

by Poincare duality Q_M is non-degenerate (it is a pairing)

$H^2(M,\mathbb{Z})$ is a free abelian group

Q_M is diagonalisable, the signature of Q_M is defined as for bilinear forms on real vector space, $\operatorname{sign}(M) = \#$ positive numbers $- \#$ negative values

intersection form it carries interesting information about M

used by Freedman to classify all simply connected 4-manifolds

used by Donaldson to prove existence of a top. 4-manifold which does not admit smooth structure

It turns out that $\operatorname{sign}(M)$ can be computed as an index of an elliptic operator of Laplace type

Let (M,g) be an oriented compact Riemannian manifold of dim $2m$

Hodge adjoint $* : \Omega^p(M) \longrightarrow \Omega^{m-p}(M)$

$$\Omega^p(M) \times \Omega^p(M) \longrightarrow \mathbb{R}, \quad \underset{\substack{\uparrow \\ \text{pairing det. by } g}}{(\omega,\eta)_g} = \int_M \omega \wedge * \eta$$

$$\Omega^p(M) \times \Omega^{m-p}(M) \to \mathbb{R}$$

$$(\omega,\eta) \longmapsto \int_M \omega \wedge \eta$$

Hodge adjoint operator $d^* = (-1)^{\circ} * d *$

$$d^* : \Omega^{p+1}(M) \to \Omega^p(M)$$

$$\Omega(M) = \bigoplus_{p=0}^{2m} \Omega^p(M)$$

$$d + d^* : \Omega(M) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \Omega(M) \otimes_{\mathbb{R}} \mathbb{C}$$

is a self-adjoint operator, elliptic operator called the Laplace-Hodge operator

$$\tau : \Omega(M) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \Omega(M) \otimes_{\mathbb{R}} \mathbb{C}$$

$$\tau(\omega) = i^{p(p-1)+m} * \omega, \quad \omega \in \Omega^p(M)$$

It turns out that $\tau^2 = +\mathrm{Id}$ and τ anticommutes with $d + d^*$, so it follows that : 1) $\Omega(M) \otimes_{\mathbb{R}} \mathbb{C} = \Omega_+(M) \oplus \Omega_-(M)$

$$\Omega_\pm(M) = \{ \text{eigenvalue } \pm 1 \text{ of } \tau \}$$

2) $d + d^* = \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}$ $\quad D : \Omega_+(M) \longrightarrow \Omega_-(M)$

$\quad D^* : \Omega_-(M) \to \Omega_+(M)$

index D = dim Ker D − dim Coker D $\in 2\mathbb{Z}$

$\parallel$ ← Atiyah-Singer formula (63')

$\frac{1}{8}$ sign(M)

↑ if M has a spin structure

follows from the fact that Ker D, Coker have a quaternionic structure, so they are complex vector spaces of even dim.

Corollary: Rohlin (52')

sign(M) is divisible by 16 provided that M is smooth, closed with spin structure

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