Notation and conventions

(M^m, g) Riemannian manifold, fixed chart $\varphi: \mathcal{U} \longrightarrow \mathbb{R}^m$ on M

$$(\varphi^{-1})^* g(x) = \sum_{i,j=1}^{m} g_{ij}(x)\, dx_i \otimes dx_j$$

where $x_1, \dots, x_m$ are coordinates, coordinate vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ on $\mathcal{U}$, dual 1-forms $dx_1, \dots, dx_m$

coefficient functions $g_{ij}(x)$ are smooth functions $x_1, \dots, x_m$,

by definition $(g_{ij}(x))_{i=1,\dots,m}^{j=1,\dots,m}$ is the Gram matrix of $g(x)$ with respect to the basis $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right\}$ of $T_x M$

In some situations it is more convenient to work with some bases of tangent spaces.

<u>Definition</u> •) Let M be a smooth manifold of dim m and $\mathcal{U}$ be an open subset of M. Then an m-tuple of vector fields $\mathbb{X}_1, \dots, \mathbb{X}_m$ defined on $\mathcal{U}$ is called a framing of $\mathcal{U}$ if for every $x \in \mathcal{U}$: $\{\mathbb{X}_1(x), \dots, \mathbb{X}_m(x)\}$ is a basis of $T_x M$.

•) If g is a Riemannian metric on M, then the framing $\mathbb{X}_1, \dots, \mathbb{X}_m$ is called an orthonormal framing if for every $x \in \mathcal{U}$: $\{\mathbb{X}_1(x), \dots, \mathbb{X}_m(x)\}$ is an orthonormal basis of $(T_x M, g_x)$.

•) Dually, a collection of 1-forms $\omega_1, \dots, \omega_m$ defined on $\mathcal{U}$ is called a coframing over $\mathcal{U}$ if $\{\omega_1(x), \dots, \omega_2(x)\}$ is a basis of $T_x^* M$ for every $x \in \mathcal{U}$.

Remark: •) Framings form the basic tool of so called Cartan's moving frame method.

•) The vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$ give a framing over the set $\mathcal{U}$.

<u>Lemma</u> Let (M, g) be a Riemannian manifold and $\varphi: \mathcal{U} \to \mathbb{R}^m$ be a chart on M. Then there is an orthonormal framing on $\mathcal{U}$.

Proof: One can apply the Gram-Schmidt algorithm to the framing $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$.

1. $\mathbb{X}_1 = \dfrac{\frac{\partial}{\partial x_1}}{\underbrace{\| \frac{\partial}{\partial x_1} \|}_{\sqrt{g_{11}}}}$

$\sqrt{g_{11}} \in \mathcal{C}^\infty$ on $\mathcal{U}$ $\Rightarrow$ $\mathbb{X}_1$ is smooth on $\mathcal{U}$

2. $\tilde{\mathbb{X}}_2 = \dfrac{\partial}{\partial x_2} - \underbrace{\dfrac{g(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})}{g(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1})}} \dfrac{\partial}{\partial x_1}$

$\dfrac{g_{21}}{g_{11}} \in \mathcal{C}^\infty$ on $\mathcal{U}$ $\Rightarrow$ $\tilde{\mathbb{X}}_2$ is smooth on $\mathcal{U}$

$\mathbb{X}_2 = \dfrac{\tilde{\mathbb{X}}_2}{\| \tilde{\mathbb{X}}_2 \|}$, $\{\mathbb{X}_1, \mathbb{X}_2\}$ ... pair of vector fields on $\mathcal{U}$ with unit length and which are orthogonal to each other

3. $\vdots$

$\square$

<u>Definition</u> If $\{\mathbb{X}_1, .., \mathbb{X}_m\}$ is a framing on an open subset $\mathcal{U}$ of M, then the uniquely 1-form

$\{\omega_1, .., \omega_m\}$ on $\mathcal{U}$

that are determined by the requirement:

$$\omega_j(\mathbb{X}_i) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j, \end{cases}$$

form so called dual coframing to $\{\mathbb{X}_1, .., \mathbb{X}_m\}$.

Remark: Note that if $\{\mathbb{X}_1, .., \mathbb{X}_m\}$ is an orthonormal framing over $\mathcal{U} \subseteq M$ and $\{\omega_1, .., \omega_m\}$ is the dual coframing, then we may write over $\mathcal{U}$

$$g = \sum_{i=1}^{m} \omega_i \otimes \omega_i .$$

In other words, the Gram matrix of g at $x \in \mathcal{U}$ is w.r. to $\{\mathbb{X}_1, .., \mathbb{X}_m\}$ is the identity matrix.

Remark Note that a framing on a manifold need exist globally. If the manifold posses a global framing (the framing exists on the whole of the manifold), then the manifold is called parallelizable. It can be shown that there some topological obstructions to the existence of global framing. There are manifolds that do not posses a global nowhere zero vector field. For example, S^{2m} has no global nowhere zero vector field.

Musical isomorphisms

Assume that V is a real vector space of dim m with inner product g. Then g induces isomorphisms

$$\flat : V \longrightarrow V^*, \quad v \longmapsto v^\flat(u) = g(v,u) = g(u,v), \quad u \in V$$

$$\# : V^* \longrightarrow V, \quad \alpha \longmapsto \alpha^\# \in V \text{ s.t. } \alpha(u) = g(\alpha^\#, u) = g(u, \alpha^\#), \quad \forall u \in V$$

Here $V^* = \{\alpha : V \longrightarrow \mathbb{R},\ \alpha \text{ linear}\}$. These two isomorphisms are inverse to each other.

Assume that (M,g) is a Riemannian manifold. Then we can define two isomorphisms (of vector bundles)

$$\flat : TM \longrightarrow T^*M, \quad T_xM \ni v \longmapsto v^\flat \in T_x^*M, \quad v^\flat(u) = g_x(u,v), u \in T_xM, \ x \in M$$

$$\# : T^*M \longrightarrow TM, \quad T_x^*M \ni \alpha \longmapsto \alpha^\# \in T_xM, \quad \alpha(u) = g_x(\alpha^\#, u), \quad \text{---''---}$$

Finally, we define

$$\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M), \quad X \longmapsto X^\flat \in \Omega^1(M), \quad X^\flat(Y) = g(X,Y), \quad Y \in \mathfrak{X}(M)$$

$$\# : \Omega^1(M) \longrightarrow \mathfrak{X}(M), \quad \omega \longmapsto \omega^\# \in \mathfrak{X}(M), \quad \omega(Y) = g(\omega^\#, Y), \quad \forall Y \in \mathfrak{X}(M)$$

It remains to show that (in each chart) $\omega^\#, X^\flat$ are smooth. Left as an exercise. By M. Berger.

Einstein notation

If $T \in \mathcal{T}^{k,l}(M)$ and φ is the chart as above, then on the domain U of φ we may write

$$T(x) = \sum_{\substack{i_1 \dots i_k \\ j_1 \dots j_l = 1}}^{m} T_{i_1 \dots i_k}^{j_1 \dots j_l}(x)\, dx_{i_1} \otimes \dots \otimes dx_{i_k} \otimes \frac{\partial}{\partial x_{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{j_l}}$$

smooth function on U

Geometers and physicists rather write only

$$T = T_{\underbrace{ab\cdots c}_{k \text{ indices}}}^{\overbrace{de\cdots f}^{\ell \text{ indices}}}$$

Operations on tensor fields

1) Symmetization

$T_{(ab\cdots c)}^{de\cdots f}$ ← symmetization in the k bottom indices

$T_{(ab)c\cdots d}^{e\cdots f}$ ... first and second

$T_{(a)b(c)\cdots d}^{e\cdots f}$ -- first and third

2) Skew-symmetization

$T_{[ab\cdots c]}^{d\cdots f}$ ← skew-sym. in all k bottom indices

3) Contraction $\quad T \in \mathcal{T}^{k,\ell}(M)$

$T_{\underline{a}b\cdots c}^{\underline{a}d\cdots e} \in \mathcal{T}^{k-1,\ell-1}(M)$

↑ contraction in the first bottom index and in the first upper index

4) Raising and lowering indices (on Riemannian manifold) ≡ musical isomorphisms

$\mathfrak{X}(M) \ni X^a \ldots \quad X \longmapsto X^\flat \in \Omega^1(M) = \mathcal{T}^{1,0}(M)$

$\|$

$\mathcal{T}^{0,1}(M)$

in Einstein notation

$$X^a \longmapsto X_a = g_{ab} X^b$$

↖ Riemannian metric

$\Omega^1(M) \ni \omega \longmapsto \omega^\# \in \mathfrak{X}(M)$

$$\omega_a \mapsto \omega^a = g^{ab}\omega_b$$

↖ the inverse of g_{ab}

5) Covariant derivative of tensor field (w.r. to affine connection ∇)

$$T \in \mathcal{T}^{k,l}(M) \longrightarrow \nabla T \in \mathcal{T}^{k+1,l}(M)$$

$$T_{ab\dots c}{}^{de\dots f} \longrightarrow \nabla_a T_{b\dots c}{}^{de\dots f}$$

Most formulas in geometry are consequences of symmetries of tensor fields. Consider:

I claim that there is no non-zero tensor field $t \in \mathcal{T}^{3,0}(M)$ which is symmetric in the first two indices and skew-symmetric in the last indices.

$$t_{abc} = t_{bac} = -t_{bca} = -t_{cba} = t_{cab} = t_{acb} = -t_{abc}$$

$$\Rightarrow \quad t_{abc} = 0$$

Fundamental Lemma of Riemannian geometry, since (using the Cartan's moving frame method) one can show the uniqueness and existence of the Levi-Civita is consequence of the previous claim.
