Exterior algebra

Let us assume that V is a real vector space of finite dimension $m \in \mathbb{N}$. Recall that $T^k V^*$ denotes the vector space of all multilinear maps

$$\underbrace{V \times \cdots \times V}_{k\text{-copies}} \longrightarrow \mathbb{R}.$$

If $B = \{v_1, \dots, v_m\}$ is a basis of V and $B^* = \{\varepsilon_1, \dots, \varepsilon_m\}$ is the dual basis, then the set

$$\{\varepsilon_I \mid I \text{ is multiindex of length } k\}$$

is a basis of $T^k V^*$. Here we use the following notation,

$I = (i_1, \dots, i_k)$ where $i_1, \dots, i_k \in \{1, \dots, m\}$ and

$\varepsilon_I = \varepsilon_{i_1 \dots i_k} = \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_k}$. We put $|I| = k$ the length of I.

In particular, $\dim T^k V^* = m^k$.

We denote by $\Lambda^k V^*$ the subspace of $T^k V^*$ of those multilinear maps which are skew-symmetric, that is, $\alpha \in \Lambda^k V^*$ if

$$\alpha(u_1, \dots, u_i, \dots, u_j, \dots, u_k) = -\alpha(u_1, \dots, u_j, \dots, u_i, \dots, u_k)$$

where $u_1, \dots, u_k \in V$. We call $\alpha \in \Lambda^k V^*$ also an alternating k-form or just k-form.

There is a canonical projection

$$\mathrm{Alt}: T^k V^* \longrightarrow \Lambda^k V^*$$

$$(\mathrm{Alt}\,\alpha)(u_1, \dots, u_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}\sigma \; \alpha(u_{\sigma(1)}, \dots, u_{\sigma(k)})$$

S_k ... the group of all permutations on k letters.

In particular if $\varepsilon_I \in T^k V^*$ is a basis element from the basis above, then we write $\varepsilon_I^\wedge := \varepsilon_{i_1} \wedge \varepsilon_{i_2} \wedge \cdots \wedge \varepsilon_{i_k} = \mathrm{Alt}(\varepsilon_I)$ where

$I = i_1, \dots, i_k$.

Now it is easy to see that the set

$$\{\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k} \mid I = (i_1, \dots, i_k) \text{ is strictly increasing}\}$$

is a basis of $\Lambda^k V^*$. In particular, $\dim \Lambda^k V^* = \binom{m}{k}$.

Wedge product on ΛV^*

Assume that $\alpha \in \Lambda^k V^*$, $\beta \in \Lambda^\ell V^*$. Then we can define

$\alpha \wedge \beta \in \Lambda^{k+\ell} V^*$ by the following composition

$$
\begin{tikzcd}
\Lambda^k V^* \otimes \Lambda^\ell V^* \arrow[r, hook] & T^k V^* \otimes T^\ell V^* \arrow[r] & T^{k+\ell} V^* \arrow[d] \\
& \Lambda^{k+\ell} V^* & \cdot \arrow[l, "\frac{(k+\ell)!}{k!\ell!} \mathrm{Alt}"']
\end{tikzcd}
$$

Explicitly

$$\alpha \wedge \beta (u_1, \dots, u_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}\sigma \; \alpha(u_{\sigma(1)}, \dots, u_{\sigma(k)}) \cdot \beta(u_{\sigma(k+1)}, \dots, u_{\sigma(k+\ell)})$$

$$= \sum_{\sigma \in Sh_{k,\ell}} \operatorname{sgn}\sigma \, \alpha(u_{\sigma(1)}, \dots, u_{\sigma(k)}) \, \beta(u_{\sigma(k+1)}, \dots, u_{\sigma(k+\ell)})$$

$Sh_{k,\ell} \subseteq S_{k+\ell}$, $\sigma \in Sh_{k,\ell}$ if we have

$\sigma(1) < \sigma(2) < \dots < \sigma(k)$ and $\sigma(k+1) < \dots < \sigma(k+\ell)$.

We call $\sigma \in Sh_{k,\ell}$ is called the shuffle of type k, ℓ.

In particular, if $k = 1$, then we may write

$$\alpha \wedge \beta (u_1, \dots, u_{k+1}) = \sum_{\sigma \in cyc} \operatorname{sgn}\sigma \; \alpha(u_{\sigma(1)}) \, \beta(u_{\sigma(2)}, \dots, u_{\sigma(k+1)})$$

where cyc is the subgroup of S_{k+1} generated by $(2, 3, \dots, k+1, 1)$.

<u>Lemma</u> If $\alpha \in \Lambda^k V^*$, $\beta \in \Lambda^\ell V^*$, then

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha .$$

Proof:

$$\alpha \wedge \beta (u_1, \dots, u_{k+\ell}) = \beta \wedge \alpha (u_{k+1}, \dots, u_{k+\ell}, u_1, \dots, u_k)$$

$$\| $$

$$(-1)^\ell \alpha \wedge \beta (u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_{k+\ell}, u_k)$$

$$\| $$

$$(-1)^\ell (-1)^\ell \alpha \wedge \beta (u_1, \dots, u_{k-2}, u_{k+1}, \dots, u_{k+\ell}, u_{k-1}, u_k)$$

$$\vdots$$

$$\underbrace{(-1)^\ell \cdots (-1)^\ell}_{k\text{-copies}} \alpha \wedge \beta (u_{k+1}, \dots, u_{k+\ell}, u_1, u_2, \dots, u_k) = (-1)^{k\ell} \alpha \wedge \beta (u_{k+1}, \dots, u_{k+\ell}, u_1, \dots, u_k) \qquad \square$$

Let $\Lambda(V^*) = \bigoplus_{k=0}^{m} \Lambda^k V^*$, here $\Lambda^0 V^* = \mathbb{R}$.

Observation $(\Lambda(V^*), \wedge)$ is a graded algebra.

Hodge star operator

Assume that the vector space V comes with an inner product $\langle -,- \rangle_V$. Recall the previous lecture that there are musical isomorphisms $\flat : V \longrightarrow V^*$

$\# : V^* \longrightarrow V$.

These two isomorphisms are inverse to each other. In particular, there is a unique inner product on V^* for which these linear isomorphisms are isometries. If B is an orthonormal basis of V, then B^* is an orthonormal basis of V^*.

$\langle -,- \rangle_V$ induces an inner product $\langle \, , \rangle_{T^kV^*}$ on $T^k V^*$ (and $\Lambda^k V^*$). This inner product satisfies (or is completely determined by)

$$\langle \alpha_1 \otimes \cdots \otimes \alpha_k, \beta_1 \otimes \cdots \otimes \beta_k \rangle_{T^kV^*} = \langle \alpha_1, \beta_1 \rangle_{V^*} \cdot \langle \alpha_2, \beta_2 \rangle_{V^*} \cdots \langle \alpha_k, \beta_k \rangle_{V^*}$$

where $\alpha_i, \beta_i \in V^*$. In particular, it follows that the basis $\{ \varepsilon_I \mid |I| = k \}$ is an orthonormal basis of $T^k V^*$.

Now we can restrict $\langle -,- \rangle_{T^kV^*}$ from T^kV^* to $\Lambda^k V^*$; we get an inner product $\langle -,- \rangle_{\Lambda^k V^*}$ on $\Lambda^k V^*$. We have that

$$\langle \alpha, \beta \rangle_{\Lambda^k V^*} = \det \left(\langle \alpha_i, \beta_j \rangle_{V^*} \right)_{i,j = 1,\dots,k}$$

where $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$, $\beta = \beta_1 \wedge \cdots \wedge \beta_k$, $\alpha_i, \beta_i \in V^*$, $i = 1, \dots, k$.

From this it follows

Lemma The basis $\{ \varepsilon_I^\wedge \mid |I| = k, I \text{ is strictly increasing} \}$ is orthonormal w.r. to $\langle -,- \rangle_{\Lambda^k V^*}$.

Finally, assume that we have fixed an orientation on V and thus also on V^*. Then we get a canonical isomorphism (which depends however on our choices)

$$\Lambda^m V^* \longrightarrow \mathbb{R}, \quad \alpha_1 \wedge \dots \wedge \alpha_m \longmapsto \det(\alpha_1 | \dots | \alpha_m)$$

where $(\alpha_1 | \dots | \alpha_m) \in M_{m\times m}(\mathbb{R})$ is the matrix whose i-th column is the vector of coefficients of α_i with respect to any positively oriented orthonormal basis of V^* (w.r. to $\langle -,- \rangle$). Here $\alpha_i \in V^*$, $i = 1, \dots, m$.

Note that if B is positively oriented and orthonormal, then $B^* = \{\varepsilon_1, \dots, \varepsilon_m\}$ is also positively oriented and orthonormal and

$$dV := \varepsilon_1 \wedge \varepsilon_2 \wedge \dots \wedge \varepsilon_m \longmapsto 1.$$

dV is some canonical element in $\Lambda^m V^*$, sometimes called the volume associated to $\langle -,- \rangle_V$ and the given orientation.

Let us consider the following pairing

$$\Lambda^k V^* \times \Lambda^{m-k} V^* \longrightarrow \Lambda^m V^* \cong \mathbb{R}$$

$$(\alpha, \beta) \longmapsto \alpha \wedge \beta.$$

This pairing is nondegenerate and so it induces map

$$(\square) \qquad * : \Lambda^{m-k} V^* \longrightarrow (\Lambda^k V^*)^* \xrightarrow{\sim} \Lambda^k V^*$$

by duality or using adjoint map construction. (Nondegeneracy means that for every fixed $0 \neq \beta \in \Lambda^{m-k} V^*$, the map

$$\Lambda^k V^* \longrightarrow \Lambda^m V^*, \quad \alpha \longmapsto \alpha \wedge \beta \quad \text{is non-zero.})$$

Explicitly, the map $(\square)$ is determined by

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle_{\Lambda^{m-k} V^*} \, dV, \quad \alpha \in \Lambda^{m-k} V^*.$$

<u>Definition</u> The map defined for every $k = 0, \dots, m$

$$* : \Lambda^{m-k} V^* \longrightarrow \Lambda^k V^*$$

is called the Hodge star operator.

Examples: •) $V = \mathbb{R}^2$ with the Euclidean inner product and canonical orientation, $\{\varepsilon_1, \varepsilon_2\}$ is dual to the canonical basis of $\mathbb{R}^2$, then $dV = \varepsilon_1 \wedge \varepsilon_2$ and so

$*\varepsilon_1 = \varepsilon_2 \qquad * \varepsilon_2 = -\varepsilon_1$

•) $V = \mathbb{R}^3$ with canonical choices

$*\varepsilon_1 = \varepsilon_2 \wedge \varepsilon_3, \; *\varepsilon_2 = -\varepsilon_1 \wedge \varepsilon_3, \; *\varepsilon_3 = \varepsilon_1 \wedge \varepsilon_2$

$*\, \varepsilon_2 \wedge \varepsilon_3 \in \Lambda^1 \mathbb{R}^{3*}, \; \alpha \in \Lambda^2 \mathbb{R}^{3*}$

$$\alpha \wedge *(\varepsilon_2 \wedge \varepsilon_3) = \langle \alpha, \varepsilon_2 \wedge \varepsilon_3 \rangle \, dV$$

$\alpha = a\, \varepsilon_1 \wedge \varepsilon_2 + \underline{b}\, \varepsilon_2 \wedge \varepsilon_3 + c\, \varepsilon_1 \wedge \varepsilon_3$

$* \varepsilon_2 \wedge \varepsilon_3 = e\, \varepsilon_1 + f \varepsilon_2 + g\, \varepsilon_3$

$(\underline{a}\, \varepsilon_1 \wedge \varepsilon_2 + \underline{b}\, \varepsilon_2 \wedge \varepsilon_3 + \underline{c}\, \varepsilon_1 \wedge \varepsilon_3) \wedge (\underline{e}\varepsilon_1 + \underline{f}\varepsilon_2 + \underline{g}\varepsilon_3)$

$= (ag + be - cf)\, \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$

$\langle \alpha, \varepsilon_2 \wedge \varepsilon_3 \rangle = b$

$ag + be - cf = b$ for $\forall a, b, c \Rightarrow g = 0, e = 1, f = 0$

$\boxed{* \varepsilon_2 \wedge \varepsilon_3 = \varepsilon_1}$

$\varepsilon_I^\wedge \in \Lambda^{m-k} V^*$

<u>Lemma</u> Let $\{\varepsilon_1, \dots, \varepsilon_m\}$ be the positively oriented and orthonormal basis of V^* as above. Then

1) $* \varepsilon_I^\wedge = \operatorname{sgn} \sigma \, \varepsilon_{I^c}^\wedge$

where $I = (i_1, \dots, i_{m-k})$ is a multiindex of length $m-k$ and I^c is the multiindex of length k which can be described by omitting $i_1, \dots, i_{m-k}$ from $(1, \dots, m)$ and σ is the permutation $(i_1, \dots, i_{m-k}, j_1, \dots, j_k)^{-1}$ where $I^c = (j_1, \dots, j_k)$.

2) $** = (-1)^{k(m-k)} \mathrm{Id}$ on $\Lambda^k V^*$.

Proof: Let us first prove the first claim. By definition,
$\varepsilon_I^\wedge \wedge * \varepsilon_I^\wedge = \langle \varepsilon_I, \varepsilon_I \rangle \, dV = dV$. On the other hand
$\varepsilon_I^\wedge \wedge \varepsilon_{I^c}^\wedge = \varepsilon_{i_1 \dots i_{m-k}} \wedge \varepsilon_{j_1 \dots j_k} = \operatorname{sgn}\sigma \; dV$.

Here $\varepsilon_{i_1 \dots i_{m-k}} = \varepsilon_{i_1} \wedge \dots \wedge \varepsilon_{i_{m-k}}$. Now the claim easily follows since $\{ \varepsilon_I^\wedge \mid I$ is an increasing multiindex of length $m-k \}$ is an orthonormal basis of $\Lambda^{m-k} V^*$.

Let us prove the second claim. Consider

$$\Lambda^k V^* \xrightarrow{*} \Lambda^{m-k} V^* \xrightarrow{*} \Lambda^k V^*.$$

By definition, $* \varepsilon_I^\wedge$ is determined by

$$\varepsilon_I^\wedge \wedge * \varepsilon_I^\wedge = dV \quad (I \text{ is an multiindex of length } k).$$

So we have that $* \varepsilon_I^\wedge \wedge ** \varepsilon_I^\wedge = dV$.

On the other hand

$$dV = \varepsilon_I^\wedge \wedge * \varepsilon_I^\wedge \overset{\text{by Lemma from last wk}}{=} (-1)^{k(m-k)} * \varepsilon_I^\wedge \wedge \varepsilon_I^\wedge =$$

$$= * \varepsilon_I^\wedge \wedge (-1)^{k(m-k)} \varepsilon_I^\wedge .$$

Again using the fact that $\{ \varepsilon_I^\wedge \mid \dots \}$ is an orthonormal basis of $\Lambda^k V^*$, we have that

$$(-1)^{k(m-k)} \varepsilon_I^\wedge = ** \varepsilon_I^\wedge .$$

In other words, $*^2 = (-1)^{k(m-k)} \, Id$. $\square$

Examples: 1) Observe that $k=2, m=4$, we have that

$$* : \Lambda^2 \mathbb{R}^4 \longrightarrow \Lambda^2 \mathbb{R}^4, \quad *^2 = (-1)^{2 \cdot 2} Id = Id.$$

We see that $*$ is an involution and

$$(D) \quad \Lambda^2 \mathbb{R}^4 = \Lambda^2 \mathbb{R}^4_+ \oplus \Lambda^2 \mathbb{R}^4_-$$

where $\Lambda^2 \mathbb{R}^4_\pm$ is the subspace of $\Lambda^2 \mathbb{R}^4$ with eigenvalue ± 1 of $*$. It turns out that these two subspaces have same dim.

$$\binom{4}{2} = 6 = 3+3, \quad \dim \Lambda^2 \mathbb{R}^4_\pm = 3.$$

(D) is a decomposition into so-called self-dual and anti-self-dual forms. Important in geometry of 4-manifolds.

2) Volume form dS^m on sphere of radius $R > 0$ in $\mathbb{R}^{m+1}$.

Consider polar coordinates on $\mathbb{R}^{m+1}$ $(r, \underbrace{\theta_1, \dots, \theta_m}_{\text{spherical coordinates}})$

r: distance from the origin

The corresponding differential form dr has the property that $dr \wedge dS^m = d\lambda^{m+1}$ ← the Lebesgue measure on $\mathbb{R}^{m+1}$

This means $dS^m = * dr$. In standard coordinates on $\mathbb{R}^{m+1}$ we have that

$$dr = d\left(\sqrt{x_1^2+\dots+x_{m+1}^2}\right) = \frac{x_1 dx_1 + \dots + x_{m+1} dx_{m+1}}{\underbrace{\sqrt{x_1^2+\dots+x_{m+1}^2}}_{|x|}}$$

and so $* dr$ is equal to

$$\Omega = \frac{x_1 * dx_1}{|x|} + \frac{x_2}{|x|} * dx_2 + \dots + \frac{x_3}{|x|} * dx_{m+1}.$$

Hence, if $\iota : S^m \hookrightarrow \mathbb{R}^{m+1}$ then

$$\iota^* \Omega = dS^m.$$

If $m=2$, then $\Omega = \frac{x_1 dx_2}{|x|} - \frac{x_2 dx_1}{|x|}$.

If I rewrite Ω back in polar coordinates

$x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$

$$\Omega = \frac{r\cos\varphi \, d(r\sin\varphi) - r\sin\varphi \, d(r\cos\varphi)}{r}$$

$$= \frac{r\cos\varphi(\sin\varphi \, dr + r\cos\varphi \, d\varphi) - r\sin\varphi(\cos\varphi \, dr - \sin\varphi \, r \, d\varphi)}{r}$$

$$= \frac{\cancel{\cos\varphi\sin\varphi \, dr} + r^2\cos^2\varphi \, d\varphi - \cancel{\sin\varphi\cos\varphi \, dr} + r^2\sin^2\varphi \, d\varphi}{r}$$

$$= r\,d\varphi.$$

If we pullback to S^1 with radius R and integrate, then indeed

$$\int_0^{2\pi} R\,d\varphi = 2\pi R.$$