Exterior derivative

Let M be a smooth manifold of dim m and $x \in M$. Let $\Lambda^k T_x^* M$ be the vector space of all skew-symmetric multilinear maps

$$T_x M \times \dots \times T_x M \longrightarrow \mathbb{R}, \quad k = 0, \dots, m.$$

Then $\bigcup_{x \in M} \Lambda^k T_x^* M =: \Lambda^k T^* M$ has a canonical smooth atlas and so it is a smooth manifold of dimension $m + \binom{m}{k}$ such that the canonical projection

$$\pi : \Lambda^k T^* M \to M \text{ is smooth.}$$

It is clear that $\Lambda^k T^* M \subset T^{(k,0)} M$.

<u>Definition</u> A k-form (or a smooth k-form) on the manifold M is a smooth map $\omega : M \longrightarrow \Lambda^k T^* M$ such that $\pi \circ \omega = Id_M$.

(Alternatively, a k-form on M is $\omega \in \mathcal{T}^{(k,0)} M$ such that for every $x \in M$;

$$\omega_x : \underbrace{T_x M \times \dots \times T_x M}_{k\text{-copies}} \longrightarrow \mathbb{R}$$

is skew-symmetric.) We denote by $\Omega^k(M)$ the space of all k-forms on M.

Remark: It is clear that $\Omega^k(M)$ is (infinite dimensional) vector space.

Local description of k-forms in chart

Let $\varphi : U \to \mathbb{R}^m$ be a chart on M with coordinate vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ and dual forms $dx_1, \dots, dx_m$. We will use the following convention, if $I = (i_1, \dots, i_k)$

$$dx_I = dx_{i_1 \dots i_k} = dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

is k-form on U which is determined by that fact that it is skew-symmetric and $dx_{i_1} \wedge \dots \wedge dx_{i_k} \left(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_k}} \right) = 1$.

Then any k-form on $\mathcal{U}$ (or more precisely the restriction of any $\omega \in \Omega^k(M)$ to $\mathcal{U}$) can be written in a unique way as

$$\omega_x = \sum_{I : |I| = k}' f_I(x)\, dx_I$$

where $\sum'$ denotes that the summation is performed only over strictly increasing multiindeces. Here f_I are smooth functions on $\mathcal{U}$.

Wedge product operation on

$$\Lambda^k T_x^* M \times \Lambda^\ell T_x^* M \to \Lambda^{k+\ell} T_x^* M$$

for any $x \in M$ gives rise to

$$\Lambda^k T^* M \times \Lambda^\ell T^* M \longrightarrow \Lambda^{k+\ell} T^* M$$

and also

$$\Omega^k(M) \times \Omega^\ell(M) \longrightarrow \Omega^{k+\ell}(M),$$

$$(\omega, \eta) \longmapsto (\omega \wedge \eta)_x (u_1, \dots, u_{k+\ell})$$

$$= \omega_x \wedge \eta_x (u_1, \dots, u_{k+\ell})$$

$$= \sum_{\sigma \in Sh(k,\ell)} \operatorname{sgn} \sigma \; \omega_x(u_{\sigma(1)}, \dots, u_{\sigma(k)})\, \eta_x(u_{\sigma(k+1)}, \dots, u_{\sigma(k+\ell)}),$$

$u_1, \dots, u_{k+\ell} \in T_x M$, $x \in M$.

Let us recall from the winter semester that if $f : M \to N$ is a smooth map of manifolds, then there is the associated pullback map

$$f^* : \mathcal{T}^{(k,0)} N \to \mathcal{T}^{(k,0)} M \quad , \; k = 0, 1, \dots\dots$$

It is easy to verify that if we restrict f^* to $\Omega^k(N)$, then we get a map

$$\Omega^k(N) \longrightarrow \Omega^k(M)$$

which we also denote by f^*.

The map f^* is in coordinates given by:

$$f^*\omega = f^*\Big({\sum_{I:|I|=k}}' g_I \, dy_I \Big) =$$

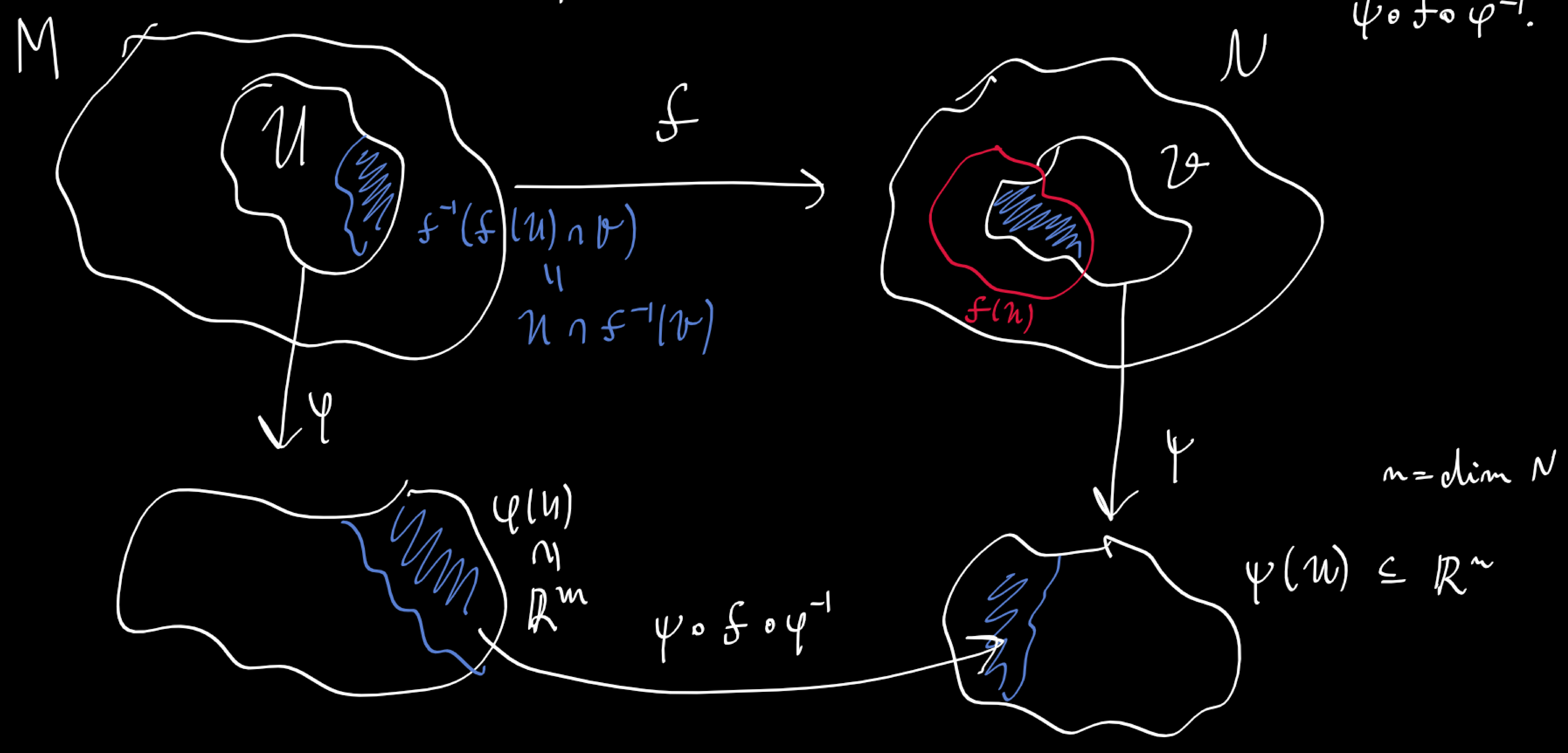
$$= {\sum_{I:|I|=k}}' (g_I \circ f) \, f^* dy_I$$

where $\psi : \mathcal{V} \to \mathbb{R}^n$ is a chart on N with coordinates $y_1, \dots, y_n$ and $g_I \circ f$ is just composition of functions and

$$f^* dy_I = f^*(dy_{i_1} \wedge \dots \wedge dy_{i_k})$$
$$= (f^* dy_{i_1}) \wedge (f^* dy_{i_2}) \wedge \dots \wedge (f^* dy_{i_k}),$$

$I = (i_1, \dots, i_k)$,

$$f^* dy_j = \sum_{l=1}^{m} \frac{\partial f^j}{\partial x_l} dx_l,$$ f^j is the j-th component of $\psi \circ f \circ \varphi^{-1}$.



$y_j = f^j(x_1, \dots, x_m)$

$j = 1, \dots, n,$

Abstract definition of exterior derivative

Theorem There exists a unique linear map
$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M), \quad k = 0,\dots, m-1, \quad m = \dim M,$$
with the following properties:

(ED1) df is the exterior derivative of $f \in \mathcal{C}^\infty(M) = \Omega^0(M)$, that is, $df = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i} dx_i$ in any chart on M.

(ED2) $d \circ d = 0$ (in other words
$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M)$$
is a complex)

(ED3) $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta$
where $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$.

Proof of uniqueness of d: Assume that $\varphi : U \to \mathbb{R}^m$ and
$$\omega = \sum_{I : |I| = k}{}' f_I \, dx_I \quad \text{is as above.}$$

Then
$$d\omega = d\Big(\sum_I{}' f_I \, dx_I \Big) = \sum_I{}' d(\underset{\substack{\uparrow \\ 0\text{-form on } U}}{f_I} \, \underset{\substack{\uparrow \\ k\text{-form on } U}}{dx_I})$$

$$= \sum_I{}' df_I \wedge dx_I + \sum_I{}' (-1)^0 f_I \wedge ddx_I = \circledast$$

$ddx_I = 0$ by (ED2) since
$$dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k} = d(x_{i_1} dx_{i_2} \wedge \dots \wedge dx_{i_k})$$

$$\circledast = \sum_I{}' df_I \wedge dx_I = \underbrace{\sum_I{}' \sum_{j=1}^{m} \frac{\partial f}{\partial x_j} dx_j \wedge dx_I}_{\text{local formula for } d}.$$