Coordinate-free formula for exterior derivative

Let $\omega \in \Omega^k(M)$ where M is a smooth manifold of dim m.
Consider

$$d'\omega(X_0,\dots,X_k) = \sum_{i=0}^{k} (-1)^i X_i\big(\omega(X_0,\dots,\hat{X}_i,\dots,X_k)\big)$$

$$+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k)$$

where $X_0,\dots,X_k \in \mathfrak{X}(M)$ and $\wedge$ means omission. We see that the right side is a smooth function on M and so

$$d'\omega : \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{k+1 \text{ copies}} \longrightarrow \mathcal{C}^\infty(M)$$

<u>Lemma</u> The map $d'\omega$ is an element of $\Omega^{k+1}(M)$.

Proof: 1) $d'\omega$ is linear $\mathcal{C}^\infty(M)$, that is

$$d'\omega(X_0,\dots,fX_i,\dots,X_k) = f\, d'\omega(X_0,\dots,X_k)$$

where $f \in \mathcal{C}^\infty(M)$ and $i = 0,\dots,k$ are arbitrary. If we prove this, then $d'\omega \in \mathcal{T}^{k+1,0}(M)$.

$$d'\omega(X_0,\dots,fX_i,\dots) = \sum_{i<j}^{k} (-1)^j X_j\big(\omega(X_0,\dots,fX_i,\dots,\hat{X}_j,\dots)\big)$$

$$+ \sum_{j<i} (-1)^j X_j\big(\omega(X_0,\dots,\hat{X}_j,\dots,fX_i,\dots)\big)$$

$$+ (-1)^i fX_i\, \omega(X_0,\dots,\hat{X}_i,\dots\dots)$$

$$+ \sum_{\substack{0 \le j < l \le k \\ j,l \ne i}} (-1)^{j+l} \omega([X_j,X_l],X_0,\dots,\hat{X}_j,\dots,\hat{X}_l,\dots,X_k)$$

$$+ \sum_{i<j\le k} (-1)^{i+j} \omega([fX_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots)$$

$$+ \sum_{0\le j<i} (-1)^{j+i} \omega([X_j,fX_i],X_0,\dots,\hat{X}_j,\dots,\hat{X}_i,\dots) =$$

$$= \sum_{j=0}^{k} (-1)^j f X_j (\omega(X_0, \dots, X_i, \dots, \hat{X}_j, \dots))$$
$$+ \sum_{\substack{j=0 \\ j \neq i}}^{k} (-1)^j (X_j f) \omega(X_0, \dots, \hat{X}_j, \dots)$$
$$+ \sum_{0 \leq j < l \leq k} (-1)^{j+l} f \omega([X_j, X_l], \dots, \hat{X}_j, \dots, \hat{X}_j, \dots X_k)$$
$$+ \sum_{i < j \leq k} (-1)^{i+j+1} (X_j f) \omega(X_i, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k)$$
$$+ \sum_{0 \leq j < i} (-1)^{j+i} (X_j f) \omega(X_i, \dots, \hat{X}_j, \dots, \hat{X}_i, \dots, X_k)$$
$$= \sum_{j=0}^{k} (-1)^j f X_j (\omega(X_0, \dots, X_i, \dots, \hat{X}_j, \dots))$$
$$+ \sum_{0 \leq j < l \leq k} (-1)^{j+l} f \omega([X_j, X_l], \dots, \hat{X}_j, \dots, \hat{X}_j, \dots X_k)$$
$$\left.\begin{aligned} &+ \sum_{\substack{j=0 \\ j \neq i}}^{k} (-1)^j (X_j f) \omega(X_0, \dots, \hat{X}_j, \dots) \\ &+ \sum_{i < j \leq k} (-1)^{j+1} (X_j f) \omega(X_0, \dots, X_i, \dots, \hat{X}_j, \dots, X_k) \\ &+ \sum_{0 \leq j < i} (-1)^{j+1} (X_j f) \omega(X_0, \dots, \hat{X}_j, \dots, X_i, \dots, X_k) \end{aligned}\right\} = 0$$
$$= f d'\omega(X_0, \dots, X_k).$$

We have shown that $d'\omega \in \mathcal{T}^{k+1,0}(M)$.

2) Remains to show that $d'\omega$ is skew-symmetric.
Fix $i < j$ with $0 \leq i, j \leq k$. Then we have that

$$d'\omega(\mathbb{X}_0,\dots,\underset{\substack{\uparrow\\ i\text{-th}}}{\mathbb{X}_j},\dots,\underset{\substack{\uparrow\\ j\text{-th position}}}{\mathbb{X}_i},\dots,\mathbb{X}_k) =$$

$$= \sum_{l=0}^{k}(-1)^l \mathbb{X}_l\, \omega(\mathbb{X}_0,\dots,\mathbb{X}_j,\dots,\hat{\mathbb{X}}_l,\dots,\mathbb{X}_i,\dots,\mathbb{X}_k)$$

$$+ \sum_{0\le p<q\le k}(-1)^{p+q}\omega([\mathbb{X}_p,\mathbb{X}_q],\mathbb{X}_0,\dots,\mathbb{X}_j,\dots,\hat{\mathbb{X}}_p,\dots,\hat{\mathbb{X}}_q,\dots,\mathbb{X}_i,\dots,\mathbb{X}_k)$$

$$= -\sum_{l=0}^{k}(-1)^l \mathbb{X}_l\, \omega(\mathbb{X}_0,\dots,\mathbb{X}_i,\dots,\hat{\mathbb{X}}_l,\dots,\mathbb{X}_j,\dots,\mathbb{X}_k)$$

$$- \sum_{0\le p<q\le k}(-1)^{p+q}\omega([\mathbb{X}_p,\mathbb{X}_q],\mathbb{X}_0,\dots,\mathbb{X}_i,\dots,\hat{\mathbb{X}}_p,\dots,\hat{\mathbb{X}}_q,\dots,\mathbb{X}_j,\dots,\mathbb{X}_k)$$

$$= -d\omega'(\mathbb{X}_0,\dots,\mathbb{X}_i,\dots,\mathbb{X}_j,\dots,\mathbb{X}_k).$$

$\square$

<u>Lemma</u> Let $\omega\in\Omega^k(M)$. Then $d'\omega = d\omega\in\Omega^{k+1}(M)$

Proof: Assume $\varphi: U\to\mathbb{R}^m$ is a chart on M and that

$$\omega = \sum_{I:\,|I|=k}{}' f_I\, dx_I.$$

Then since d' is clearly linear, then it is enough to verify that

$$d'(f_I\,dx_I) = df_I\, dx_I$$

for any fixed increasing multiindex $I=(i_1,\dots,i_k)$.
Since we know that $d'(f_I\,dx_I)$ is a $(k+1)$-form in U, then it is enough to verify that

$$(d' f_I\,dx_I)\left(\frac{\partial}{\partial x_{j_0}},\dots,\frac{\partial}{\partial x_{j_k}}\right) = d(f_I\,dx_I)\left(\frac{\partial}{\partial x_{j_0}},\dots,\frac{\partial}{\partial x_{j_k}}\right)$$

for every increasing multiindex $(j_0,\dots,j_k)=J$.

By definition,

$$(d' f_I \, dx_I)\left(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_k}}\right) =$$

$$= \sum_{l=0}^{k} (-1)^l \frac{\partial}{\partial x_{j_l}} f_I \underbrace{dx_{i_1} \wedge \dots \wedge dx_{i_k}\left(\frac{\partial}{\partial x_{j_0}}, \dots, \widehat{\frac{\partial}{\partial x_{j_l}}}, \dots, \frac{\partial}{\partial x_{j_k}}\right)}$$

recall $\left[\frac{\partial}{\partial x_{j_l}}, \frac{\partial}{\partial x_{j_m}}\right] = 0$

$$= \boxed{\sum_{l=0}^{m} (-1)^l \frac{\partial}{\partial x_{j_l}} f_{j_0 \dots j_{l-1} j_{l+1} \dots j_k}}$$

"1" precisely when

$i_1 = j_0$ $\quad i_l = j_{l+1}$
$i_2 = j_1$ $\quad \vdots$
$\vdots$ $\quad i_k = j_k$
$i_{l-1} = j_{l-1}$

Otherwise it is $= 0$

On the other hand

$$d(f_I \, dx_I) = \sum_{l=1}^{m} \frac{\partial}{\partial x_{i_l}} f_I \, dx_{i_l} \wedge dx_I$$

$$d(f_I \, dx_I)\left(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_k}}\right) = \boxed{\sum_{n=0}^{m} (-1)^n \frac{\partial}{\partial x_{j_n}} f_{j_0 \dots j_{n-1} j_{n+1} \dots j_k}}$$

$$(dx_{i_l} \wedge dx_I)\left(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_k}}\right) \begin{cases} \neq 0 & \text{if } (i_l, i_1, \dots, i_k) \text{ is a permutation } (j_0, \dots, j_k) \\ = 0 & \text{otherwise} \end{cases}$$

if $(i_l, i_1, \dots, i_k)$ is a permutation of $(j_0, \dots, j_k)$, then since $(i_1, \dots, i_k)$ and $(j_0, \dots, j_k)$ are strictly increasing, then $i_l = j_n$ for some $n = 0, \dots, k$ and then we need precisely n transpositions to move $(i_l, i_1, \dots, i_k)$ to $(j_0, \dots, j_k)$, hence we may write $(i_l, i_1, \dots, i_k) = (j_n, j_0, \dots, \hat{j}_n, \dots, j_k)$. $\square$

We have also proved existence of d since $d'\omega \in \Omega^{k+1}(M)$. This completes the proof of uniqueness and existence of d. $\square$

Remark:

1) Note that if $\omega \in \Omega^1(M)$, then

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]), \quad X, Y \in \mathfrak{X}(M).$$

2) Cartan formula, if $\omega \in \Omega^k(M)$, $X \in \mathfrak{X}(M)$ and $i_X : \Omega^\ell(M) \longrightarrow \Omega^{\ell-1}(M)$, $(i_X \theta)(X_1, \dots, X_{\ell-1}) = \theta(X, X_1, \dots, X_{\ell-1})$ where $X_1, \dots, X_{\ell-1} \in \mathfrak{X}(M)$. Then

$$\mathcal{L}_X \omega = i_X d\omega + d i_X \omega.$$

Theorem (Naturality of d)

Let $f: M \longrightarrow N$ be a smooth map of manifolds. Then d intertwines with the pullback of f, that is, for any k the following digram commutes

$$\begin{array}{ccc} \Omega^k(N) & \xrightarrow{d} & \Omega^{k+1}(N) \\ \downarrow f^* & & \downarrow f^* \\ \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M). \end{array}$$

Proof: Left as an exercise. $\square$

Theorem (Stokes' formula)

Let M be a smooth oriented manifold of dimension m with boundary ∂M. Then

$$\int_M d\omega = \int_{\partial M} \omega, \quad \omega \in \Omega^{m-1}(M).$$

(Where ∂M has a canonical orientation.)

Proof: Recall the course Introduction into analysis of manifolds. $\square$

Examples 1) $f = f_1 dx_1 + f_2 dx_2 \in \Omega^1(\mathbb{R}^2)$, then

$$df = df_1 \wedge dx_1 + df_2 \wedge dx_2 = \left(\frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2\right) \wedge dx_1 + \left(\frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2\right) \wedge dx_2 = \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx_2.$$

2) $\Omega^0(\mathbb{R}^3) \xrightarrow{d=\nabla} \Omega^1(\mathbb{R}^3) \xrightarrow{d=\mathrm{curl}} \Omega^2(\mathbb{R}^3) \xrightarrow{d=\mathrm{div}} \Omega^3(\mathbb{R}^3)$

$$\overset{\psi}{f} \longmapsto df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 \rightsquigarrow \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) = \nabla f$$

$$f = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$$

$$df = \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx_2 + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right) dx_1 \wedge dx_3$$

$$f = f_{12}\, dx_1 \wedge dx_2 + f_1\, dx_2 \wedge dx_3 + f_{13}\, dx_1 \wedge dx_3$$

$$df = \left(\frac{\partial f_{12}}{\partial x_3} + \frac{\partial f_{23}}{\partial x_1} - \frac{\partial f_{13}}{\partial x_2}\right) dx_1 \wedge dx_2 \wedge dx_3$$