Hodge star operator

Let (M, g) be a Riemannian manifold of dimension m. Then we know that g induces metric on each vector space

(1) $\Lambda^k T^*_x M = \{ f : T_x \times \dots \times T_x M \to \mathbb{R} \mid f$ is multilinear and skew-symmetric $\}$

and it also induces isomorphism (if we assume that $\Lambda^m T^*_x M$ is an oriented vector space)

(2) $$\Lambda^m T^*_x M \simeq \mathbb{R}$$

for every $x \in M$. The isomorphism (2) gives rise to so called volume form ω_g on M which allows to compute integrals of functions on M ("integral of the first kind"). In coordinate chart with coordinate functions $x_1, \dots, x_m$ we have that

$$\omega_g = \sqrt{\det(g_{ij}(x))} \; dx_1 \wedge \dots \wedge dx_m$$

where $g(x) = \sum_{i,j=1}^{m} g_{ij}(x)\, dx_i \otimes dx_j$. However note that $\omega_g \notin \Omega^m(M)$ unless we assume that (M, g) is oriented.

Let us assume that (M, g) is oriented. Then we have the Hodge star operator $*$ for every $x \in M$:

(3) $$* : \Lambda^k T^*_x M \longrightarrow \Lambda^{m-k} T^*_x M, \quad k = 0, \dots, m.$$

This linear operator is defined by the following requirement

$$\eta \wedge * \varphi = \langle \eta, \varphi \rangle_g \, \omega_g(x)$$

where $\eta, \varphi \in \Lambda^k T^*_x M$, $\langle -, - \rangle_g$ is the inner product on $\Lambda^k T^*_x M$ induced by g. The linear operator (3) extends also to

(4) $$* : \Omega^k(M) \longrightarrow \Omega^{m-k}(M), \quad k = 0, \dots, m$$

$$(* \omega)_x = *(\omega_x), \quad x \in M, \; \omega \in \Omega^k(M).$$

Let us recall that

$$** = (-1)^{k(m-k)} \, \mathrm{Id}.$$

Let us for a moment assume that M is compact and let $\omega, \theta \in \Omega^k(M)$. Then from the definition of $*$ (in particular using its compatibility with ω_g) it follows that

$$\langle\langle \omega, \theta \rangle\rangle = \int_M \langle \omega, \theta \rangle_g \, \omega_g = \int_M \omega \wedge * \theta$$

holds for every $\omega, \theta \in \Omega^k(M)$. If M is not compact, then we have to work the Hilbert space of square integrable forms on M with measurable coficients, that is,

$$\|\omega\|^2 = \int_M \langle \omega, \omega \rangle_g \omega_g < +\infty.$$

Let us now assume that $\omega, \theta \in \Omega^k(M)$ and that

$$\|\omega\| < +\infty, \quad \|\theta\| < +\infty.$$

Then also

$$|\langle\langle \omega, \theta \rangle\rangle| = \left| \int_M \langle \omega, \theta \rangle_g \, \omega_g \right| < +\infty.$$

<u>Definition</u>

An operator

$$\delta = d^*: \mathcal{D}^k(M) \longrightarrow \Omega^{k-1}(M)$$

is a formal adjoint to

$$d: \Omega^{k-1}(M) \longrightarrow \Omega^k(M)$$

if

$$\langle\langle d\omega, \eta \rangle\rangle = \langle\langle \omega, \delta\eta \rangle\rangle$$

for every $\eta \in \mathcal{D}^k(M)$, $\omega \in \Omega^{k-1}(M)$ and $\mathcal{D}^k(M)$ is the subspace of $\Omega^k(M)$ of forms with compact support. ($\eta \in \mathcal{D}^k(M)$ is $\eta = 0$ outside a compact subset $K \subseteq M$.)

Remark: It can be shown that δ exists and is unique.

<u>Theorem</u> The formally adjoint operator

$$d^* = \delta: \mathcal{D}^k(M) \longrightarrow \Omega^{k-1}(M)$$

to d is $\delta = d^* = (-1)^{m(k-1)-1} * d *$.

$$\Omega^k(M) \xrightarrow{*} \Omega^{m-k}(M) \xrightarrow{d} \Omega^{m-k+1}(M) \xrightarrow{*} \Omega^{k-1}(M)$$

Proof: Assume that $\eta \in \mathcal{D}^k(M)$ and that $\omega \in \Omega^{k-1}(M)$.
Then by Stoke's theorem

$$\int_M d(\omega \wedge * \eta) = \int_{\partial M} \omega \wedge * \eta = 0.$$

On the other hand

$$\int_M d(\omega \wedge * \eta) = \int_M (d\omega) \wedge * \eta + (-1)^{k-1} \int_M \omega \wedge d * \eta$$

$$\int_M d\omega \wedge * \eta = \langle\langle d\omega, \eta \rangle\rangle \quad \text{by the previous slide}$$

$$\int \omega \wedge d * \eta = \int \omega \wedge * * (-1)^{\ell} d * \eta = \langle\langle \omega, * (-1)^{\ell} d * \eta \rangle\rangle.$$

$$\eta \in \Omega^k(M), \quad * \eta \in \Omega^{m-k}(M), \quad d * \eta \in \Omega^{m-k+1}(M)$$

$$** = (-1)^{\ell} \, Id_{\Omega^{m-k+1}(M)}$$

$$\ell = (m-k+1)(m-(m-k+1)) = (m-k+1)(k-1)$$

We have proved that

$$\langle\langle d\omega, \eta \rangle\rangle = \langle\langle \omega, * (-1)^{k-1+\ell+1} d * \eta \rangle\rangle, \quad \omega \in \Omega^{k-1}(M), \ \eta \in \mathcal{D}^k(M).$$

$$\delta = (-1)^{k+\ell} * d * . \qquad \square$$

Formally adjoint operator in coordinates

We will consider only $k=1$. If $\eta \in \mathcal{D}^1(M)$, then using partition of unity underlying atlas on M (which means that if $\mathcal{A} = \{ \varphi_\alpha : U_\alpha \to M \mid \alpha \in A \}$ is atlas on M then we have a collection $x_\alpha : M \to [0,1]$ such that

$$x_\alpha \in \mathcal{C}^\infty(M) \text{ with compact support in } U_\alpha \text{ and}$$

$$\sum_{\alpha \in A} x_\alpha = 1.$$ We are assuming that $\{U_\alpha, \alpha \in A\}$ is locally finite.) we can write η is a locally finite sum of forms with support in each chart U_α, more explicitly

$\eta = \sum_{\alpha \in A} x_\alpha \cdot \eta = \sum_{\alpha \in A} \eta_\alpha$ where $\eta_\alpha \in \mathcal{D}^1(M)$ with support contained in U_α.

Hence without loss of generality we may assume that the support of η is contained in $\mathcal{U}_\alpha$ and compute the formal adjoint δ for such a 1-form. Then

$$\delta\eta = \delta \sum_{\alpha\in A} \eta_\alpha = \sum_{\alpha\in A} \delta\eta_\alpha .$$

Let $\varphi : \mathcal{U} \to M$ be a chart on M with coordinate function $x_1, \dots, x_m$, with coordinate fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$. Assume that

$$g(x) = \sum_{i,j=1}^{m} g_{ij}(x)\, dx_i \otimes dx_j .$$

Recall that g induces isomorphisms

$$\flat : TM \longrightarrow T^*M, \quad T_xM \ni v \longmapsto v^\flat \in T_x^*M$$

$$v^\flat(u) = g(v,u) = g(u,v)$$

$$u \in T_xM,\ x\in M.$$

If we apply $\flat$ in the coordinate chart φ, then $\frac{\partial}{\partial x_i}^\flat$ is a smooth 1-form in $\mathcal{U}$ defined by

$$\omega_i = \frac{\partial}{\partial x_i}^\flat, \quad \omega_i(x) = \sum_{j=1}^{m} g_{ij}(x)\, dx_j .$$

Let us recall that the induced metric on T^*M is determined by the fact that $\flat$ is an isometry for every $x\in M$. Hence we can now compute this bundle metric $\langle -,- \rangle_g$ in the chart φ w.r. to the basis $dx_1, \dots, dx_m$. We have that

$$\sum_{i=1}^{m} g^{ij}(x)\, \omega_i(x) = \sum_{l,i=1}^{m} \underbrace{g^{ij}(x)\, g_{il}(x)}_{\delta_{jl}}\, dx_l = dx_j .$$

We have that

$$dx_j = \sum_{i=1}^{m} g^{ij}(x)\, \omega_i(x), \quad j=1,\dots,m.$$

Hence

$$\langle dx_i, dx_j \rangle_g = \sum_{n,l=1}^{m} g^{li}(x)\, g_{ln}(x)\, g^{nj}(x)$$

$$= \sum_{l,m=1}^{m} g^{ij}(x) .$$

Now we can compute the formally adjoint operator δ for $k=1$ in the chart φ. Recall that

$$\langle\langle df, \eta \rangle\rangle = \langle\langle f, \delta\eta \rangle\rangle, \quad f \in C^\infty(M).$$

The left hand side is the chart φ given by

$$\int_U \sum_{i,j=1}^m \frac{\partial f}{\partial x_i} g^{ij}(x)\, a_j(x) \sqrt{\det(g_{ij})}\, dx_1 \wedge \dots \wedge dx_m =$$

Stokes theorem $\Big| \; \eta = \sum_{i=1}^m a_i(x) dx_i, \quad a_i \in C^\infty(U) \; \Big|$

$\downarrow$

$$= - \int_U \sum_{i,j=1}^m f \frac{\partial}{\partial x_i}\left(g^{ij}(x) a_j(x) \sqrt{\det g(x)} \right) dx_1 \wedge \dots \wedge dx_m$$

The right side is

$$\int_U f \,\delta\Big(\sum_{i=1}^m a_i(x)\, dx_i \Big) \sqrt{\det g}\, dx_1 \wedge \dots \wedge dx_m$$

Since these two formulas have to agree for every f, η, we have that

$$\text{(FA)} \quad \boxed{\delta\Big(\sum_{i=1}^m a_i \, dx_i \Big) = - \frac{1}{\sqrt{\det g(x)}} \sum_{i,j=1}^m \frac{\partial}{\partial x_i}\left(g^{ij}(x) a_j(x) \sqrt{\det g(x)} \right).}$$

Now if $\eta = X^\flat$ for X smooth vector field with

$$X = \sum_{j=1}^m b_j \frac{\partial}{\partial x_j} \quad \text{so that}$$

$$\eta = \sum_{i=1}^m \underbrace{\sum_{j=1}^m g_{ij} b_j}_{a_i} dx_i, \quad \text{then (FA) reduces to}$$

$$\delta(X^\flat) = - \frac{1}{\sqrt{\det g(x)}} \sum_i \frac{\partial}{\partial x_i}\left(b_i(x) \sqrt{\det g(x)} \right) =$$

$$\Big| \; \sum_j g^{ij} a_j = \sum_{i,j} g^{ij} g_{j\ell} b_\ell = b_i \; \Big|$$

$$= -\frac{1}{\sqrt{\det g}} \left(\sum_{i=1}^{m} \frac{\partial a_i}{\partial x_i} \sqrt{\det g} + a_i \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \sqrt{\det g} \right)$$

$$= -\left(\sum_{i=1}^{m} \frac{\partial a_i}{\partial x_i} + \frac{a_i}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \sqrt{\det g} \right) = \operatorname{div} X$$

is the formula for the divergence operator on vector fields.