Short recapitulation from last week

(M,g) is a Riemannian oriented manifold of dimension m, then the Hodge star operator induces a linear operator

$$*: \Omega^k(M) \longrightarrow \Omega^{m-k}(M)$$

We have defined $\delta = d^* = * \circ d \circ * \, (-1)^{m(k+1)-1}$

$$\Omega^k(M) \xrightarrow{*} \Omega^{m-k}(M) \xrightarrow{d} \Omega^{m-k+1}(M) \xrightarrow{*} \Omega^{k-1}(M),$$

then by Stokes's theorem

$$\langle\langle df, g \rangle\rangle = \langle\langle f, d^* g \rangle\rangle$$

for any $f \in \Omega^{k-1}(M)$, $g \in \mathcal{D}^k(M)$. In other words, d^* is the formal adjoint adjoint to d.

By Stokes's theorem, we have found that for $k=1$ the formal adjoint operator $d^* = \delta$ is given by the formula

$$\delta\Big(\underbrace{\sum_{i=1}^m a_i \, dx_i}_{\text{1-form with compact}}\Big) = -\frac{1}{\sqrt{\det g(x)}} \sum_{i,j=1}^m \frac{\partial}{\partial x_i}\Big(g^{ij}(x) a_j(x) \sqrt{\det g(x)}\Big).$$

support in the domain of a chart $\varphi: U \to \mathbb{R}^m$. In the case that X is a vector field on M with compact support with coordinate functions $a_1, \dots, a_m$ in U, then $X^\flat$ is a 1-form

$$-\left(\sum_{i=1}^m \frac{\partial a_i}{\partial x_i} + \frac{a_i}{\sqrt{\det g}} \frac{\partial}{\partial x_i}\sqrt{\det g}\right) = \operatorname{div} X$$

<u>Definition</u> The operator $\operatorname{div}: \mathfrak{X}(M) \longrightarrow C^\infty(M)$ defined by the formula above is called the divergence operator.

<u>Theorem</u> Let (M,g) be an oriented Riemannian manifold with the ass. volume form $\omega_g \in \Omega^m(M)$. Then

$$\mathcal{L}_X \omega_g = -\operatorname{div} X \cdot \omega_g.$$

Proof: By Cartan's formula

$$\mathcal{L}_{X}\,\omega_g = (d i_X + i_X d)\,\omega_g = d i_X \omega_g =$$

$$= d\Big(i_{\sum_{l=1}^m a_l \frac{\partial}{\partial x_l}} \sqrt{\det(g_{ij})}\, dx_1\wedge\dots\wedge dx_m\Big)$$

$$= d\Big(\sum_{l=1}^m (-1)^{l-1} a_l \sqrt{\det(g_{ij})}\, dx_1\wedge\dots\wedge\widehat{dx_l}\wedge\dots\wedge dx_m\Big)$$

$$= \sum_{k=1}^m \frac{\partial}{\partial x_k}\Big(\sum_{l=1}^m (-1)^{l-1} a_l \sqrt{\det(g_{ij})}\Big)\, \underbrace{dx_k\wedge\dots\wedge\widehat{dx_l}\wedge\dots\wedge dx_m}_{l=k}$$

$$= \sum_{l=1}^m \frac{\partial}{\partial x_l}\Big((-1)^{l-1} a_l \sqrt{\det(g_{ij})}\Big)\, dx_l\wedge dx_1\wedge\dots\wedge\widehat{dx_l}\wedge\dots\wedge dx_m$$

$$= \underbrace{\sum_{l=1}^m \frac{\partial}{\partial x_l}\Big(a_l\sqrt{\det(g_{ij})}\Big)\frac{1}{\sqrt{\det g}}}_{-\operatorname{div} X}\,\omega_g$$

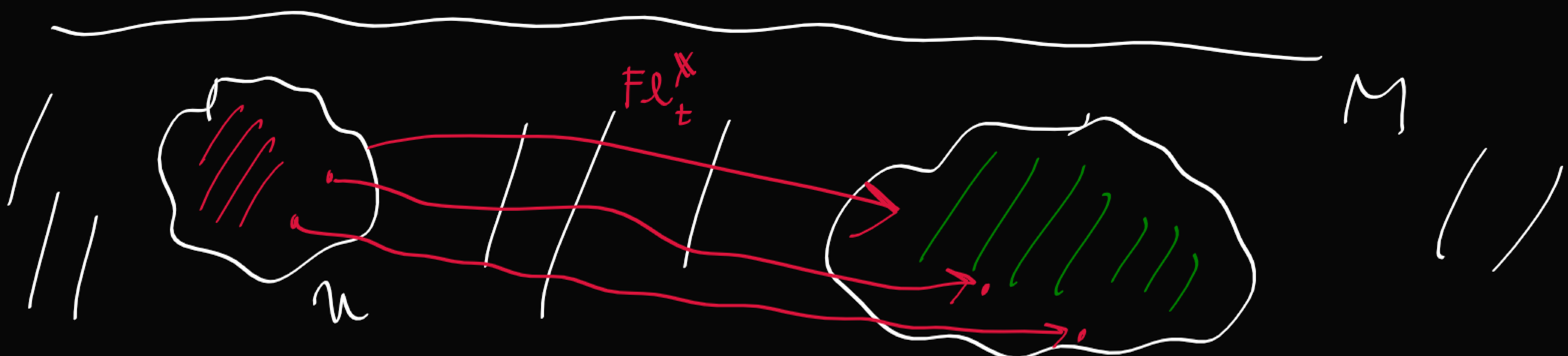
$$- \operatorname{div} X .$$

$\square$

Remark: The previous Theorem provides a geometric meaning to the divergence operator. It is not hard to show that $\operatorname{div} X = 0$ iff the flow of X preserves the measure (w.r. to ω_g) of open subsets of M, that is the measure of any $U \subseteq M$ is the same as the measure of

$$Fl_t^X(U) \subseteq M \text{ for any } t \text{ for which}$$

$Fl_t^X(U)$ is well defined, iff Fl_t^X is the flow of incompressible fluid.



Example If (M, g) is $(\mathbb{R}^m, g_{st})$, $\quad g_{st}$ ↑ Euclidean metric on $\mathbb{R}^m$

then

$$\operatorname{div} X = -\sum_{i=1}^m \frac{\partial a_i}{\partial x_i} \qquad g_{st} = \sum_{i=1}^m dx_i \otimes dx_i$$

if $X = \sum_{i=1}^m a_i \frac{\partial}{\partial x_i}$, $a_i \in \mathcal{C}^\infty(\mathbb{R}^m)$.

Laplace – Beltrami operator

Definition Let (M, g) be a Riemannian manifold. Then the operator

$$\Delta : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$$
$$\Delta f = \delta \circ d f = \operatorname{div} \circ (df)^\#$$

is called the Laplace – Beltrami operator.

In coordinate chart $\varphi : U \to \mathbb{R}^m$ as above, we have that

$$\Delta f = -\frac{1}{\sqrt{\det g}} \sum_{i=1}^m \frac{\partial}{\partial x_i} \left(\sum_{j=1}^m \frac{\partial f}{\partial x_j} g^{ij} \sqrt{\det g} \right)$$

with the same notation and conventions as before.

Assume that $f \in \mathcal{C}^\infty(M)$, $h \in \mathcal{D}^1(M)$, then we know that

$$\int_M \langle df, h \rangle_g \, \omega_g = \langle\langle df, h \rangle\rangle = \langle\langle f, \delta h \rangle\rangle = \int_M f \, \delta h \, \omega_g$$

where $\langle -,- \rangle_g$ is the bundle metric on T^*M induced by g.

Theorem Δ is formally self-adjoint, that

$$\langle\langle \Delta f, h \rangle\rangle = \langle\langle f, \Delta h \rangle\rangle$$

whenever f or $h \in \mathcal{D}(M)$.

Proof: We have

$$\langle\langle \Delta f, h \rangle\rangle = \langle\langle \delta \circ d f, h \rangle\rangle = \langle\langle df, dh \rangle\rangle = \langle\langle f, \delta dh \rangle\rangle$$
$$= \langle\langle f, \Delta h \rangle\rangle . \qquad \square$$

Theorem (Spectrum of Laplace–Beltrami operator)
The spectrum of Δ is real and non-negative, that if M is closed, $f \in \mathcal{C}^\infty(M)$ and $\Delta f = \lambda f$ for some $\lambda \in \mathbb{R}$, then
$$\lambda \geq 0.$$

Proof:
$$\langle\langle f, \Delta f\rangle\rangle = \langle\langle f, \delta \circ df\rangle\rangle = \langle\langle df, df\rangle\rangle = \|f\|^2 \geq 0$$
$$\langle\langle f, \lambda f\rangle\rangle = \lambda \|f\|^2. \quad \Rightarrow \quad \lambda \in \mathbb{R} \text{ and } \lambda \geq 0. \qquad \square$$

(Alternatively one can consider $f \in \mathcal{D}(M)$ if M is non-compact or $f = 0$ on ∂M if M is compact.)

Theorem (Naturality of Δ)
Let (M, g_M) and (N, g_N) are Riemannian manifolds and that
$$\Phi : M \longrightarrow N$$
is a local isometry. Then
$$\Delta_M \Phi^* f = \Phi^* \Delta_N f, \quad f \in \mathcal{C}^\infty(N).$$

Proof: This follows from the fact that $\Delta = \delta \circ d$ and that d and δ commute with Φ^*. Alternatively, one can verify this any coordinate chart. $\square$

Example: $(M, g_M) = (\mathbb{R}^m, g_{st})$ then
$$\Delta f = -\sum_{i=1}^{m} \frac{\partial^2 f}{\partial x_i^2}.$$