Spectrum of Δ on spheres

(M,g) is closed oriented Riemannian manifold, the spectrum $S(\Delta)$ of the Laplace-Beltrami operator Δ is a subset of $[0,+\infty)$, this subset is disrete and unbounded. We know also that $0 \in S(\Delta)$ as a constant function c is an eigen vector of Δ as $\Delta c = d^* dc = 0$. We also know that the set of eigenvectors of Δ forms a complete orthonormal basis of $L^2(M)$.

The unit sphere S^1 in $\mathbb{R}^2$

The standard atlas on S^1 is

$$\mathcal{A}_{S^1} = \{ \varphi_a^{-1} , a \in \mathbb{R} \}$$

where

$$\varphi_a : (a, a+2\pi) \longrightarrow \mathbb{R}^2$$

$$\varphi_a(t) = (\cos t, \sin t).$$

The standard (round) Riemannian metric on S^1 is $dt \otimes dt$, so that the canonical projection

$$\mathbb{R} \longrightarrow S^1, \quad t \longmapsto (\cos t, \sin t)$$

is a local isometry.

Hence $\Delta_{S^1} = -\frac{\partial^2}{\partial t^2}$ is the Laplace - Beltrami operator on S^1.

Thus the equation for an eigenfunction for Δ_{S^1}

$$(*) \quad -\frac{\partial^2}{\partial t^2} f = \Delta_{S^1} f = \lambda f$$

together with boundary condition is that

$$\lim_{t \to a^+} f(t) = \lim_{t \to a+2\pi^-} f(t).$$

Equivalently, if we view f as the corresponding 2π-periodic function on $\mathbb{R}$, then $(*)$ is simply

$$(*)' \quad -\frac{\partial^2 f}{\partial t^2} = \lambda f, \quad f(t) = f(t+2\pi).$$

We know very well from calculus that the solutions of $(*)'$ are $\lambda = k^2 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and

$$f(t) = a \sin kt + b \cos kt, \quad a, b \in \mathbb{R}.$$

Theorem

The spectrum of Δ_{S^1} is $\{k^2 : k \in \mathbb{N}_0\}$ and the eigenspace of Δ_{S^1} with eigenvalue k^2 has dimension 2 and $\{\cos kt, \sin kt \,;\, t \in \mathbb{R}\}$ is a basis of this eigenspace.

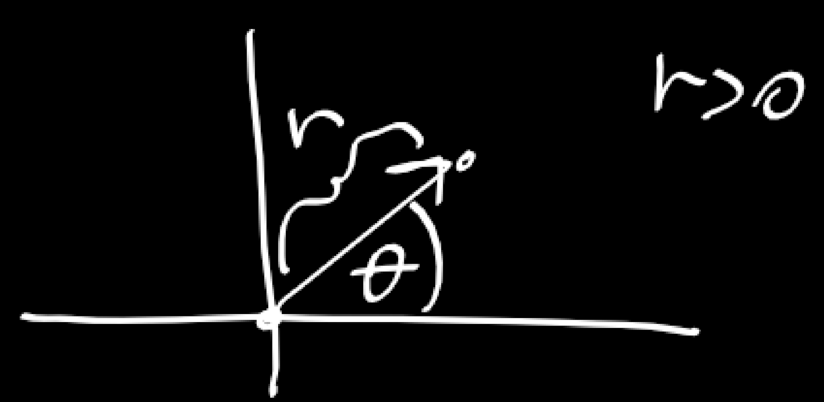
We see that the Fourier series of 2π-periodic functions on $\mathbb{R}$ is with respect to the orthonormal basis of Δ_{S^1}.

Correspondence between eigenvectors of Δ_{S^1} and harmonic function on $\mathbb{R}^3$

On $\mathbb{R}^2$ we consider the standard Euclidean metric

$$g_{Euc} = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 .$$

So that $$\Delta_2 = -\sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2} .$$



In polar coordinates,

$$-\Delta_2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\underbrace{\frac{\partial^2 f}{\partial \theta^2}}_{-\Delta_{S^1}} , \quad f \in \mathbb{R}^2.$$

Assume that $-\Delta_2 f = 0$ and that

$$f(r,\theta) = g(r)\, h(\theta) = \underbrace{r^\alpha h(\theta)}_{\mathbb{R}^n\setminus\{0\}}$$ extends to $\mathbb{R}^n$ by Theorem on removability of an isolated sing. of harmonic function

$$-\Delta_2 f = \left(g''(r) + \frac{1}{r} g'(r)\right) h(\theta) - \frac{1}{r^2} g(r)\, \Delta_{S^1} h .$$

Assume that h is an eigenvector of Δ_{S^1} with eigenvalue k^2, $k \in \mathbb{N}_0$. Then we get that

$$-\Delta_2 f = \left(g''(r) + \frac{1}{r} g'(r)\right) h(\theta) - \frac{1}{r^2} g(r)\, k^2 h(\theta)$$

$$= \underbrace{\left(g''(r) + \frac{1}{r} g'(r) - \frac{k^2}{r^2} g(r)\right)}_{=0} h(\theta).$$

Ansatz: $g(r) = r^\alpha$, then

$$g''(r) + \frac{1}{r} g'(r) - \frac{k^2}{r^2} g(r) = 0$$

consider $g(r) = r^\alpha$, $\alpha \in \mathbb{R}$ then

$$\begin{aligned} 0 &= \alpha(\alpha-1) r^{\alpha-2} + \frac{1}{r} \alpha r^{\alpha-1} - \frac{k^2}{r^2} r^\alpha \\ &= \alpha(\alpha-1) r^{\alpha-2} + \alpha r^{\alpha-2} - k^2 r^{\alpha-2} \\ &= (\alpha(\alpha-1) + \alpha - k^2) r^{\alpha-2} \\ &= (\alpha^2 - \alpha + \alpha - k^2) r^{\alpha-2} \\ &= (\alpha^2 - k^2) r^{\alpha-2} \end{aligned}$$

Solution if $|\alpha| = |k|$. Assume that $\alpha \geq 0$.

We see that:

Theorem Any eigenvector of Δ_{S^1} gives rise to a null solution of Δ_2, that is, to a harmonic function on $\mathbb{R}^2$.

Definition A polynomial p in m-variables is homogeneous of degree $k \in \mathbb{N}_0$ if
$$p(tx) = t^k p(x) \text{ for every } x \in \mathbb{R}^m.$$

If we take $\cos(k\theta)$ or $\sin(k\theta)$, then the previous computation shows that
$$r^k \cos(k\theta), \quad r^k \sin(k\theta)$$
is a harmonic function on $\mathbb{R}^2$. Note that
$$r^k \cos(k\theta) = |z|^k \operatorname{Re} e^{ik\theta} = \operatorname{Re} z^k \quad \boxed{z^k = (|z| e^{i\theta})^k = |z|^k e^{ik\theta}}$$
if we write $x+iy = z = |z| e^{i \operatorname{Arg} z} = |z| e^{i\theta}$, $|z| = r$. Similarly
$$r^k \sin(k\theta) = \operatorname{Im} z^k.$$
It follows that $r^k \cos(k\theta)$, $r^k \sin(k\theta)$ are harmonic polynomials which are homogeneous of degree k.

$\operatorname{Re} z^k$ is homogen. of degree k as a polyn. in x, y.

$\operatorname{Re}(x+iy) = x$, $\operatorname{Re}(x+iy)^2 = x^2 - y^2$, $\operatorname{Re}(x+iy)^3 = x^3 - 3xy^2, \ldots$

<u>Theorem</u> There is a canonical linear isomorphism between the eigenspace of eigenvalue k^2 of Δ_{S^1} and the vector space of harmonic polynomials on $\mathbb{R}^2$ which are homogeneous of degree k.

<u>Remark</u> If f is a harmonic function in an open set $U \subseteq \mathbb{R}^{m+1}$, then it can be proved that f is real analytic and it is equal on an open neighborhood of any point to its Taylor series which is a converging sum of harmonic homogeneous polynomials.

Correspondence between eigenvectors of Δ_{S^1} and harmonic function on $\mathbb{R}^{m+1}$, $m \geq 2$

The Laplace operator on $\mathbb{R}^{m+1}$ is in the spherical coordinates given by

$$-\Delta_{m+1} = \frac{\partial^2}{\partial r^2} + \frac{m}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_{S^m}.$$

Repeating the arguments given before, we find that

<u>Theorem</u> The spectrum of Δ_{S^m} is

$$S(\Delta_{S^m}) = \{k(k+m-1) : k = 0, 1, \dots\}$$

and the dimension of the eigenspace of the eigenvalue $k(k+m-1)$ coincides with the dimension of the vector space of harmonic polynomials in $\mathbb{R}^{m+1}$ which are hom. of degree k.

Let $\mathcal{H}_k$ be the vector space of harmonic polynomials of degree k in $\mathbb{R}^{m+1}$. Then then if $f \in \mathcal{H}_k$, then

$f|_{S^m}$ is an eigenvector of Δ_{S^m} with eigenvalue $k(k+m-1)$. The proof follows from the case $m = 1$, where we need Theorem on removability of isol. sing. of harmonic function (as for holomorphic functions).

Fisher decomposition: Let $\mathcal{P}_k$ be the vector space of hom. polynomials of degree k in $\mathbb{R}^{m+1}$. Then $\mathcal{P}_{k+2} = \mathcal{H}_{k+2} \oplus r^2 \cdot \mathcal{P}_k \Rightarrow \dim \mathcal{H}_{k+2} = \dim \mathcal{P}_{k+2} - \dim \mathcal{P}_k$
$= \binom{m+1+k+2-1}{k+2} - \binom{m+1+k-1}{k}$.

Let (M,g) be a closed oriented Rie. man. with the Laplace–Beltrami operator Δ_M. Consider $N(\lambda) : \mathbb{R} \longrightarrow \mathbb{N}$

$$N_M(\lambda) = \# \{ \mu : \mu \leq \lambda, \ \mu \in \underbrace{S(\Delta_M)}_{\text{the spectrum of } \Delta_M} \},$$

where we also count the multiplicity of each eigenvalue.

For example:

$$N_{S^1}(\lambda) = \begin{cases} 2\lfloor \lambda \rfloor, & \lambda \geq 0 \\ 0, & \lambda < 0. \end{cases}$$

<u>Theorem</u> Let (M,g) be a closed oriented Riemannian manifold, then

$$N_M(\lambda) = (2\pi)^{-m} \omega_m \, Vol(g) \, \lambda^m + \mathcal{O}(\lambda^{m-1})$$

where $m = \dim M$, $Vol(g) = \int_M 1 \, \omega_g$ is the total volume of M and ω_m is the volume of unit sphere in $\mathbb{R}^{m+1}$.

This is Theorem 3.3.1 in [2].

<u>Hodge decomposition</u>

Let (M,g) be a Riem. man. which is oriented and closed. Then

$$\Delta = d^* d + d d^* : \Omega^k(M) \longrightarrow \Omega^k(M), \quad k = 0, \ldots, m = \dim M.$$

Then Δ is the Laplace–Hodge operator and

$\alpha \in \Omega^k(M)$ is called harmonic if $\Delta \alpha = 0$.

As M is compact, then for any $\omega_1, \omega_2 \in \Omega^k(M)$

$$\langle\langle \omega_1, \omega_2 \rangle\rangle = \int_M \langle \omega_1, \omega_2 \rangle_g \, \omega_g < +\infty, \quad \| \omega_1 \| = \sqrt{\langle\langle \omega_1, \omega_2 \rangle\rangle}.$$

Assume that α is harmonic, then we have that

$$0 = \langle\langle \Delta \alpha, \alpha \rangle\rangle = \langle\langle (d d^* + d^* d) \alpha, \alpha \rangle\rangle = \langle\langle d d^* \alpha, \alpha \rangle\rangle + \langle\langle d^* d \alpha, \alpha \rangle\rangle$$
$$= \langle\langle d^* \alpha, d^* \alpha \rangle\rangle + \langle\langle d\alpha, d\alpha \rangle\rangle = \| d^* \alpha \|^2 + \| d\alpha \|^2 \Rightarrow d\alpha = 0, \ d^* \alpha = 0.$$

Let $\mathcal{H}^k_\Delta(M)$ be the vector space of all harmonic k-forms on M.

<u>Theorem</u>

1) Every harmonic form is closed.

2) Every $\omega \in \Omega^k(M)$ can be written in a unique way as

$$(\nabla) \qquad \omega = \alpha + d\beta + d^*\gamma$$

where $\alpha \in \mathcal{H}^k_\Delta(M)$, $\beta \in \Omega^{k-1}(M)$, $\gamma \in \Omega^{k+1}(M)$. Moreover

$$\|\omega\| = \|\alpha\| + \|d\beta\| + \|d^*\gamma\|,$$

that is the decomposition (∇) is orthogonal w.r. to $\| \ \|$.

3) The canonical inclusion

$$\mathcal{H}^k_\Delta(M) \longrightarrow H^k_{DR}(M)$$

is an isomorphism of finite dimensional vector spaces.

<u>Corollary</u> $\dim H^k_{DR}(M) < +\infty$ if M closed (even for M compact) as $\dim \mathcal{H}^k_\Delta(M) < +\infty$ since Δ is an elliptic operator (which is Fredholm, that is, it has finite dim. kernel and cokernel) on a compact manifold.

<u>Example</u> •) Let $\mathbb{R}^2 = \mathbb{C}$ and consider $\mathcal{H}^1_\Delta(\mathbb{R}^2)$. Assume that $\alpha = f_1\,dx + f_2\,dy \in \mathcal{H}^1_\Delta(\mathbb{R}^2)$. Then

$$\left.\begin{array}{l} 0 = d\alpha \iff \dfrac{\partial f_1}{\partial y} = \dfrac{\partial f_2}{\partial x} \\[2ex] 0 = d^*\alpha \iff \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} = 0 \end{array}\right\} \iff \begin{array}{l} \dfrac{\partial f_1}{\partial y} + \dfrac{\partial(-f_2)}{\partial x} = 0 \\[2ex] \dfrac{\partial f_1}{\partial x} - \dfrac{\partial(-f_2)}{\partial y} = 0 \end{array} \iff$$

C-R equations for $f = (f_1, -f_2) : \mathbb{C} \longrightarrow \mathbb{C}$. Hence

$$\alpha \in \mathcal{H}^1_\Delta(\mathbb{R}^2) \iff f \in \mathcal{O}(\mathbb{R}^2).$$

•) Let $S^1 \times S^1 = T^2$ with the product Riemannian metric. Consider $\mathcal{H}^1_\Delta(T^2)$. We can view $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ 

The canonical projection $\mathbb{R}^2 \to T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is again a local isometry.

We can view $\alpha \in \Omega^1(T^2)$ as 1-form on $\mathbb{R}^2$ whose coefficient functions are 2π-periodic in both variables, that is,

$$\alpha = f_1\,dx + f_2\,dy \in \Omega^1(\mathbb{R}^2)$$

with $f_i(x,y) = f_i(x+2\pi, y) = f_i(x, y+2\pi)$, $x, y \in \mathbb{R}$.

Then as above, $\alpha \in \mathcal{H}^1_\Delta(T^2)$ iff

$f = (f_1, -f_2)$ is holomorphic. As f_1, f_2 are 2π-periodic

$\Rightarrow$ $f : \mathbb{C} \to \mathbb{C}$ is bounded $\Rightarrow$ f is constant by a well know theorem from complex analysis. We see that

$\alpha = a\,dx + b\,dy$, $a, b \in \mathbb{R}$. Hence

$$\mathcal{H}^1_\Delta(T^2) \cong H^1_{DR}(T^2) \cong \mathbb{R}^2.$$