Uniformization Theorem

- .) Riemannian curvature tensor
- .) Conformal geometry
- .) Complex manifolds
- .) Uniformization theorem

(M,g) is a Riemannian manifold with Riemannian curvature tensor $R = R_{abc}{}^{d} \in \mathcal{T}^{3,1}(M)$. Then the Ricci curvature is

$$Ric_{ab} = R_{acb}{}^{c} \in \mathcal{T}^{2,0}(M)$$ and

the scalar curvature is

$$S = R_{ab}\, g^{ab} \in \mathcal{T}^{0,0}(M) = \mathcal{C}^{\infty}(M).$$

The Weyl curvature W of R is the part of R which belongs to the kernel of the first contraction.
Then

$$R = W + Ric + S$$

is the decomposition of R into $O(m)$ irreducible pieces.
(For $x \in M$, then

$$(*)\ R_x : T_xM \times T_xM \times T_xM \longrightarrow T_xM$$

is multilinear map. Since g_x is a inner product on T_xM, then there is a canonical action of $O(m)$, $m = \dim M$, on the vector of all multilinear maps such as in $(*)$.)

If $m=2$, then

$$W = 0, \quad Ric = 0.$$

If $m=3$, then

$$W = 0.$$

If $m \geq 4$, then none of the pieces (W, Ric, S) has to vanish on a general manifold.

Conformal geometry

Definition A conformal structure on a manifold M is an equivalence class of Riemannian metrics where two metrics g_1 and g_2 are equivalent if $\exists\ f \in \mathcal{C}^{\infty}(M)$, $f > 0$, such that $g_2 = f\, g_1$.

The conformal structure $[g]$ ($\equiv$ the conformal class of Riemannian metric g on M) is called flat if for every $x \in M$ there exists an open neigh. U_x of x such the flat Riemannian metric in U_x is conformal to g, that is, the flat Riemannian metric in U_x differs from g by a positive function.

Example $(S^m, [g_{st}])$ is conformally flat structure since in any stereographic projection g_{st} is conformally equivalent to the Euclidean metric.

Theorem Let $(M, [g])$ be a conformal structure on manifold M with Riemannian g. Then if $m \geq 4$, then $[g]$ is flat iff $W_g = 0$. Moreover, it can be shown that the Weyl curvature does not depend on the choice of Riemannian metric in $[g]$, that

$$W_{g_1} = W_{g_2} \quad \text{if} \quad g_1, g_2 \in [g].$$

If $m = 3$, then $[g]$ is flat iff Cotton-York vanishes. (This tensor depends on third order derivatives of g.)

If $m = 2$, then $[g]$ is flat always.

The previous Theorem in $m = 2$, is equivalent to the existence of so called isothermal coordinates. Given a Riemannian manifold (M, g) with $x \in M$, then one can find a coordinate chart $\varphi: U \to \mathbb{R}^2$ with $x \in U$ such that

$$(\varphi^{-1})^* g = e^f \underbrace{(dx \otimes dx + dy \otimes dy)}_{\text{the Euclidean metric in } \mathbb{R}^2} \quad \text{for some } f \in \mathcal{E}^\infty(\varphi(U))$$

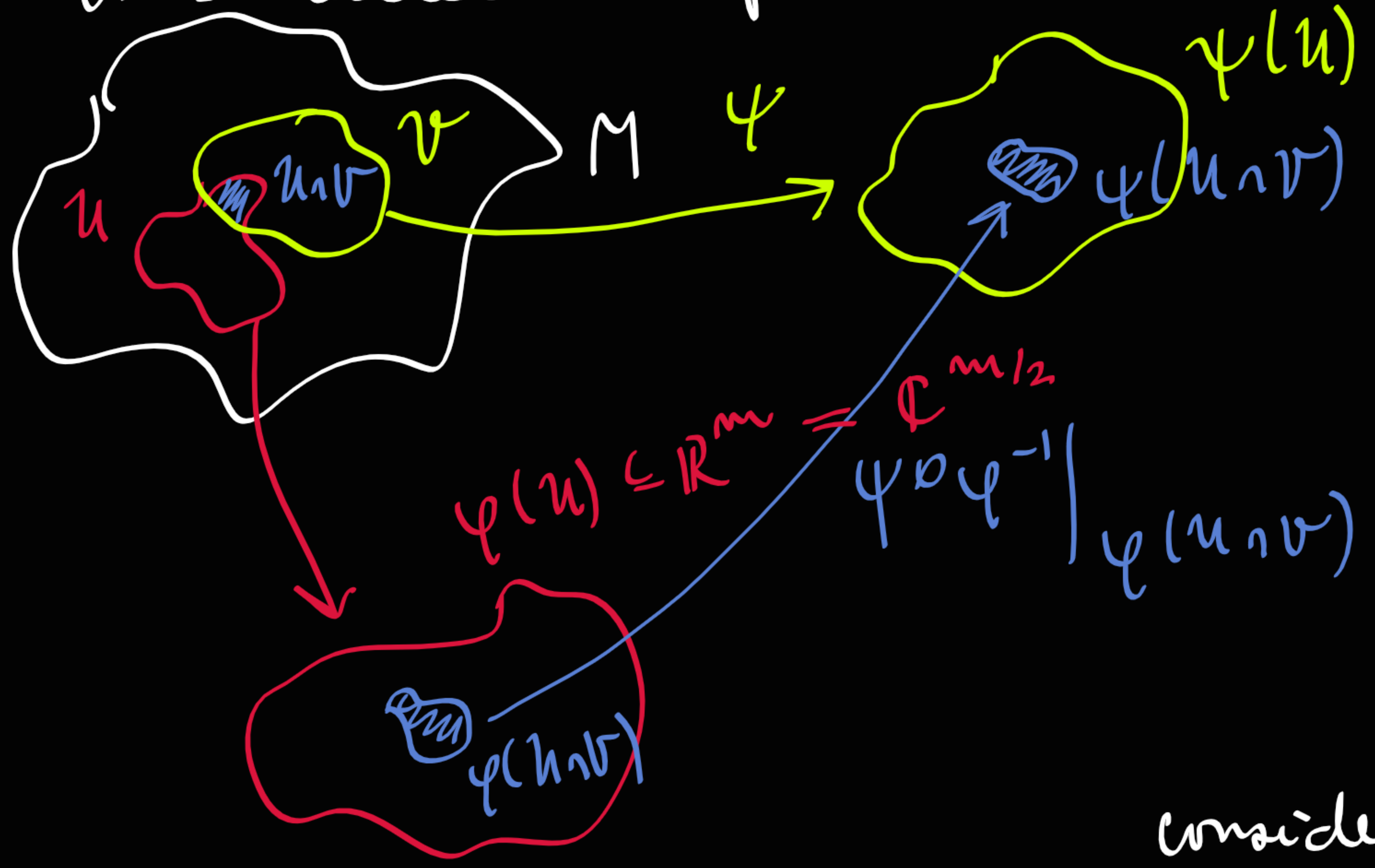
Definition A smooth map $\Phi: M \to N$ of manifolds is conformal w.r. to a Riemannian metric g_N on N and g_M on M if $\Phi^* g_N = e^f g_M$ for some $f \in \mathcal{E}^\infty(M)$.

Remark: If $g \in [g_N]$, then $\Phi^* g \in [g_M]$.

A conformal map is always a local diffeomorphism.

Complex manifolds

Definition A smooth manifold $(M, \mathcal{A})$ is complex if every transition function coming from $\mathcal{A}$ is biholomorphism. $\mathcal{A}$ is called a complex atlas.



m is even

is a biholomorphism if it is a diffeomorphism & holomorphic

consider only when $m = 2$

Any complex carries a tautological almost complex structure $J \in \mathcal{T}^{1,1}(M)$ such that

$$J_x^2 = -Id_{T_xM}, \quad x \in M.$$

J is defined in the following way:

fix $x \in M$, a chart from the complex atlas on M $\varphi : \mathcal{U} \to M$ with $x \in \mathcal{U}$, then

$$T_x\varphi : T_xM \longrightarrow T_{\varphi(x)}\,\varphi(\mathcal{U}) = T_{\varphi(x)}\,\mathbb{C}^{m/2} \cong \mathbb{C}^{m/2}.$$

$$\mathcal{U} \subseteq \mathbb{C}^{m/2}$$

We define J_x such that $T_x\varphi$ is complex linear. One can verify that: 1) J_x does not depend on the choice of φ.

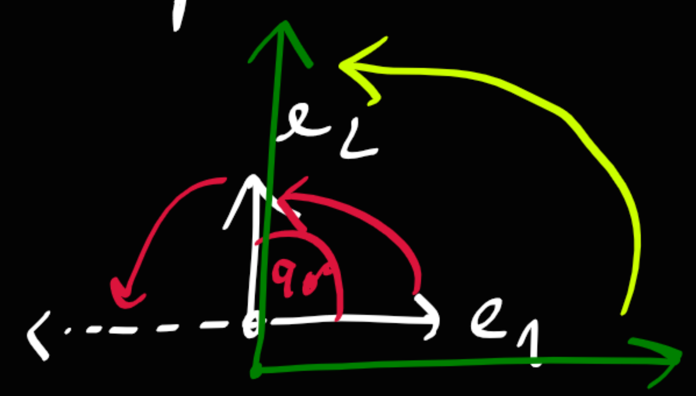
2) J_x as x ranges over M defines a global tensor field of type $(1,1)$ on M.

It can be shown that this almost complex structure is integrable, that is, the Nijenhuis tensor of J is vanishing everywhere. It can be proved that also the converse is true:
(Newlander–Nirenberg theorem)

Given a manifold M with an integrable almost complex structure J, then M is a complex manifold, that is, M admits a complex atlas such that J is the tautological almost complex structure coming from the complex atlas. (Proof in Hörmander: Analysis in several complex variables)

Complex manifolds in dimension 2 (or in complex dim. 1).

Let (M, g) be a Riemannian manifold of dim. $m = 2$. Assume that M is oriented. Now fix $x \in M$ and let (e_1, e_2) be a pos. or. orthonormal basis of $T_x M$. Define the linear map



$$J_x : T_x M \longrightarrow T_x M$$
$$J_x(e_1) = e_2, \quad J_x(e_2) = -e_1.$$

In other words, J_x is the counter-clockwise rotation in $T_x M$ by $90°$. We see that J_x does not depend on the choice of the basis. It is easy to that

$$J_x^2 = -Id_{T_x M}$$

and J_x as x varies over M defines an almost complex structure $J \in \mathcal{T}^{1,1}(M)$ on M. Note that the definition of J really depends only on $[g]$. It can be proved that J is integrable (as this is always true in dim $m = 2$). We see

<u>Theorem</u> Any 2-dimensional manifold admits a complex atlas.

Short overview from complex analysis in one variable

<u>Observation</u> Given an open subset $U \subseteq \mathbb{C}$ and a map

$\varphi : U \longrightarrow \mathbb{C}$, then then the following are equivalent:

1) φ preserves orientation (the canonical orientation) on $\mathbb{C}$ and φ is conformal (w.r.t. to the Euclidean metrics on $\mathbb{C} = \mathbb{R}^2$ and on $U \subseteq \mathbb{R}^2 = \mathbb{C}$).

2) φ is holomorphic and $\varphi' \neq 0$ on U.

Proof: Follows from the Cauchy-Riemann equation. $\square$

<u>Theorem</u> (Riemann mapping theorem)
Let $U \subsetneq \mathbb{C}$ be a simply connected and connected domain in $\mathbb{C}$. Then U is biholomorphic to the unit disc $\mathbb{D}$ in $\mathbb{C}$, that is

$$\varphi : U \xrightarrow{1-1} \mathbb{D} \text{ holomorphic.}$$

Proof is covered in Complex analysis I. $\square$

<u>Corollary</u>: Let M be a simply connected and connected Riemannian surface (2-dim. complex manifold). Then M is biholomorphic to

·) S^2 the Riemann sphere (if M compact) or

·) $\mathbb{C}$ or

·) $\mathbb{D}$.

($M \xrightarrow{1-1} \left\{ \begin{matrix} S^2 \\ \mathbb{C} \\ \mathbb{D} \end{matrix} \right.$ holomorphic).

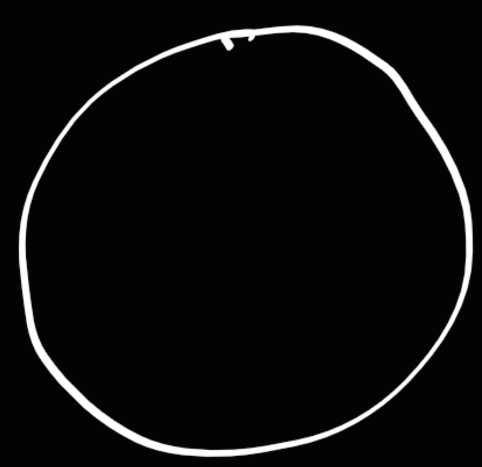
Uniformization Theorem

Theorem (Gauss-Bonnet Theorem)
Let (M, g) be a closed oriented Riemannian surface (2-dim manifold with Riemannian metric g). Then

$$2\pi \chi(M) = \int_M S_g(x)\, \omega_g(x).$$

Here $\chi(M)$ is the Euler char.

$M = S^2$



$\chi(M) = 2$ $\quad S_g > 0$

$M = S^1 \times S^1 = T^2$

$\chi(M) = 0$ $\quad S_g = 0$

$\chi(M) = 2 - 2g$ $\quad S_g < 0$

genus g .. g holes, $g > 1$

Question: Is there a metric on M with constant scalar curvature?

Theorem Let M be a simply connected and connected Riemannian surface. Then M admits a Riemannian metric with constant scalar curvature.

Proof: Let $\tilde{M}$ be the universal cover of M. Then $\tilde{M}$ is a simply connected and connected Riemannian surface, hence by the Riemann mapping theorem

$$\tilde{M} = \begin{cases} S^2 = M, & \text{round metric has scalar curvature } +1. \\ \mathbb{C}, & M = T^2 = \mathbb{R}^2/\mathbb{Z}^2, \text{ Euclidean metric on } \mathbb{R}^2 \text{ descends via } \mathbb{R}^2 \to T^2 \text{ to a metric with constant scalar curvature} \\ \mathbb{D} \cong \mathbb{H}_+, & M \text{ has genus } g \geq 1, \mid \end{cases}$$

For the surfaces of genus $g > 1$, the universal cover is biholomorphic to $\mathbb{D} \cong \mathbb{H}_+ = \{ x+iy \mid y > 0 \}$.

On $\mathbb{H}_+$ we take the hyperbolic metric

$$\frac{1}{y^2}(dx \otimes dx + dy \otimes dy) = g_{\mathbb{H}_+}.$$

We know that the scalar curvature $g_{\mathbb{H}^+}$ is -1. This metric descends via $\mathbb{H}^+ \longrightarrow M = \mathbb{H}^+ / \underset{\sim}{\Gamma}$

(torsion-free and discrete subgroup of $SL(2,\mathbb{R})$, $\mathbb{H}^+ \cong SL(2,\mathbb{R}) / SO(2)$)

to a metric on M_g with scalar curvature -1. $\square$